On the stability analysis of a mathematical model of Lassa fever disease dynamics

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Abstract

Lassa fever is a deadly disease transmitted through ingestion of food that is contaminated with infected rodent's saliva, urine or excreta, infected person and inhalation of the aerosol. In this paper, we investigated the stability analysis of the transmission dynamics of Lassa fever mathematical model. The disease-free equilibrium was shown to be both locally asymptotically stable and globally asymptotically stable, the later being shown using comparison theorem. The basic reproduction number, $R_0 < 1$, which is an important parameter in the control of Lassa fever infection, was calculated using the next generation method. We have also shown that the endemic equilibrium point, exists for $R_0 > 1$ and has been noted that this endemic equilibrium is unique and globally asymptotically stable based on Lyapunov Function. This result implies that Lassa fever disease can be totally eradicated when the basic reproduction number is less than unity. We therefore advocate for health policies that will keep the basic reproduction number below one, thereby keeping the occurrence of Lassa fever under control.

Keywords: Lassa fever, Modeling, Equilibrium points, Stability Analysis

1 Introduction

Lassa fever is an acute viral illness cause by Lassa virus. It belongs to the member of Arenavirus family. The disease was first described in the 1950s and the virus was identified in 1969, when two missionary nurses died from it in the town of Lassa in Borno state, Nigeria. Lassa fever is endemic in West African countries such as Guinea, Liberia, Sierra Leone and Nigeria [6]. Studies show that about 500, 000 cases of Lassa fever occur per year in West Africa with approximately 5000 death [8]. The animal host of the Lassa virus is the rodents called Mastomys Natalensis [3]. Currently, there is an outbreak of Lassa fever in Nigeria and 63 lives were claimed out of 212 suspected cases reported and about 17 states were affected [2].

Lassa fever can be transmitted through ingestion of food that is contaminated with infected rodent's saliva, urine or excreta, inhalation of the aerosol as occurs during the sweeping of an area where the

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droppings are present, contaminated needles or exposure to an infected aerosol in health care setting and person-to-person transmission occurs through body fluid exchange. The symptoms of Lassa fever take up to three weeks to manifest [1]. They start with a lower fever, tiredness, body ache, sore throat and headache, nausea, vomiting and even diarrhea, high fever, swelling of the face, bleeding from the eyes or nose, breathing difficulties, severe pain in the back, chest and abdomen, shock, etc. [4]. There is no US approved vaccine for Lassa fever but it can be treated using Ribavirin which is effective during early stage of infectiousness [7]. Lassa fever can be prevented by using: rodent-proof containers for food storage should be used, rodents control measures such as traps and insecticides are to be used in and around human homes, avoid eating rodent (rats), avoid attracting rodents to house by cleanliness and healthy waste disposal practices, isolation of patients till recovery is well advanced, use of gown, gloves mask and cap, careful segregation of biologically hazardous waste and sterilizing all equipment used for the patients.

This Lassa fever epidemic, with its increased frequency indicates that our current knowledge in its dynamics and public health guidelines to control the disease is not adequate [5]. The study is carried out by the use of mathematical model which is formulated based on the knowledge of Lassa fever outbreak. Hence, the present study will present a mathematical model on the spread of the Lassa fever disease dynamics with current features.

Mathematical models play a vital role in analyzing the spread and control of different diseases. In fact, epidemiological modeling helps us to understand the dynamics of these diseases so that we can trace the most important parameter(s) responsible for the disease spread [5]. It is possible to mathematically model the progress of most infectious diseases to discover the likely outcome of an epidemic or to help manage them by different control programmes. The Lassa fever disease is very dangerous and claims so many lives. Despite the several interventions currently in place, the absence of proper control measures and insufficient understanding of the natural history of the disease as well as the some features such as aerosol could provide insight into the Lassa fever epidemics. Mathematical models on the natural history and features of Lassa fever disease are lacking. It is therefore the intension of this paper to formulate a new mathematical model that will incorporates some of these features for both the human population and the vectors, with the aim of investigating the stability of the model.

2 Model formulation

The model sub-divides the total human population at time t, namely; $S_h(t)$ denoting the number of susceptible individuals, $E_h(t)$ the exposed non-infectious individuals, $I_h(t)$ denoting the number of infectious individuals (infected and diagnosed or symptomatic) and $R_h(t)$, denoting the number of recovered individuals. The total rodent (rats) population at time t, is divided into susceptible rodents, $S_r(t)$, exposed rodents, $E_r(t)$ and infectious rodents, $I_r(t)$. The susceptible human population increases due to recruitment at Λ_h, a loss of immunity from the recovered class at a rate ψ and recovered humans without immunity at a rate $(1-\upsilon)$. The susceptible human population reduces as a result of a contact with an infectious rodent and a sexual contact with infected humans, aerosols and a natural death at a rate μ_h. The exposed human population increases the transmission routes and reduces due to a transition of individuals from the class of exposed to the infected class at a rate of κ and a natural death at a rate of μ_h. The infected humans may recover with temporary immunity at a rate υ and progress to recovered class while the remaining proportion recovers without immunity and become susceptible at a rate $(1-\upsilon)$, while ϕ is the proportion of humans who recovered spontaneously. The infected human population reduces by natural death and induced death at rates μ_h and δ respectively. The recovered human population reduces by natural death at a rate μ_h and loss of immunity at a rate ψ. The susceptible rodent population increases due to recruitment at a rate Λ_r which represents the growth rate of rodents. The population of the rodents reduces upon contact with infected humans, where β_3 is the transmission probability per contact by an infected humans, natural death at a rate μ_r. The exposed rodent population increases and reduces by breakthrough into infectious class at the rate α_r, natural death at a rate μ_r. The infectious rodent population increases due to breakthrough from exposed class at a rate α_r and reduces due to natural death at a rate μ_r. Another aspect of the transmission route is the opportunistic airborne transmission-infectious that naturally cause disease by small airborne particles (aerosol) that contain microorganisms. Riley and Nardell [5] present a standard model of airborne infection usually referred to as the Wells-Riley equation. The modified version of Wells-Riley equation is used to describe airborne transmission route in this Lassa fever model. The exponent represents the degree of exposure to infection and $1-e^{-rt}$ is the probability of a single susceptible being infected. Thus, putting the above formulations and assumptions together gives the following human-rodent model, given by system of ordinary differential equations below as:

$$
\begin{aligned}
\frac{dS_h}{dt} &= \Lambda_h + \phi(1-\nu)I_h + \psi R_h - \beta_1\sigma I_r S_h - \beta_2\varepsilon I_h S_h - \eta\left(1-e^{-rt}\right)S_h - \mu_h S_h \\
\frac{dE_h}{dt} &= \beta_1\sigma I_r S_h + \beta_2\varepsilon I_h S_h + \eta\left(1-e^{-rt}\right)S_h - (\kappa+\mu_h)E_h \\
\frac{dI_h}{dt} &= \kappa E_h - \phi(1-\nu)I_h - \phi\nu I_h - (\delta+\mu_h)I_h \\
\frac{dR_h}{dt} &= \phi\nu I_h - (\psi+\mu_h)R_h \\
\frac{dS_r}{dt} &= \Lambda_r - \beta_3\vartheta I_h S_r - \mu_r S_r \\
\frac{dE_r}{dt} &= \beta_3\vartheta I_h S_r - (\alpha+\mu_r)E_r \\
\frac{dI_r}{dt} &= \alpha E_r - \mu_r I_r
\end{aligned}
\qquad (1)
$$

The associated model variables and parameters are described in Table 1.

Variable	Description
S_h	Number of Susceptible humans
E_h	Number of Exposed humans
I_h	Number of Infectious humans
R_h	Number of Recovered humans
S_r	Number of Susceptible rodents

E_r	Number of Exposed rodents
I_r	Number of Infectious rodents

Table 1: Description of the state variables of the model

Parameters	Description
Λ_h	Recruitment level of humans
Λ_r	Recruitment level of rodents
δ	Per capita Lassa-induced death rate
ψ	Recovered human loss of immunity
ϕ	Spontaneous individual recovery
β_1	Transmission rate per contact by an infectious rodent
β_2	Transmission rate per contact by an infective through sexual activity
β_3	Transmission rate per contact by an infected human
η	Relative infectiousness of individuals with aerosol
μ_h	Natural mortality rate for humans
μ_r	Natural mortality rate for rodents
κ	Progression rate of human from exposed to infected
α	Progression rate of rodents from exposed to infected
σ	Contact rate of rodent per human per unit time
ϑ	Relative human-to-rodent transmissibility of infected humans
ε	Relative human-to-human transmissibility of infected humans
r	Rate of exposure to aerosol
v	Recovery with temporary immunity

Table 2: Description of the parameters of the Lassa fever model

3 Basic properties of the Lassa fever model

3.1 Positivity and boundedness of solutions

For the Lassa fever transmission model (1) to be epidemiologically meaningful, it is important to prove that all its state variables are non-negative for all time. In other words, solutions of the model system (1) with non-negative initial data will remain non-negative for all time $t > 0$.

Theorem 1: Let the initial data be $\{(S_h(0), E_h(0), I_h(0), R_h(0), S_r(0), E_r(0), I_r(0)) \geq 0\} \in \Omega$. Then the solution set $\{S_h(t), E_h(t), I_h(t), R_h(t), S_r(t), E_r(t), I_r(t)\}$ of the system (1) is positive for all $t > 0$.

Proof: From the first equation of the model system (1) we have

$$\frac{dS_h}{dt} = \Lambda_h + \phi(1-\nu)I_h + \psi R_h - \beta_1\sigma I_r S_h - \beta_2\varepsilon I_h S_h - \eta\left(1-e^{-rt}\right)S_h - \mu_h S_h$$
$$= \Lambda_h + \phi(1-\nu)I_h + \psi R_h - \left(\beta_1\sigma I_r + \beta_2\varepsilon I_h + \eta\left(1-e^{-rt}\right) + \mu_h\right)S_h$$
$$\geq -\left(\beta_1\sigma I_r + \beta_2\varepsilon I_h + \eta\left(1-e^{-rt}\right) + \mu_h\right)S_h$$
$$\int \frac{dS_h}{S_h} \geq -\int\left(\beta_1\sigma I_r + \beta_2\varepsilon I_h + \eta\left(1-e^{-rt}\right) + \mu_h\right)dt$$
$$\Rightarrow S_h(t) \geq S_h(0)e^{-\int\left(\beta_1\sigma I_r + \beta_2\varepsilon I_h + \eta\left(1-e^{-rt}\right)+\mu_h\right)dt} \geq 0$$

It can similarly be shown that $E_h(t) > 0, I_h(t) > 0, R_h(t) > 0, S_r(t) > 0, E_r(t) > 0, I_r(t) > 0$ for all $t > 0$.

3.2 Invariant region

Theorem 2: Let $(S_h, E_h, I_h, R_h, S_r, E_r, I_r)$ be the solution of the system (1) with initial conditions and the biological feasible region $\Omega \coloneqq \Omega_h \times \Omega_r$ with $\Omega_h \coloneqq \left\{(S_h, E_h, I_h, R_h) \in \mathbb{R}_+^4 : N_h \leq \frac{\Lambda_h}{\mu_h}\right\}$ and $\Omega_r \coloneqq \left\{(S_r, E_r, I_r) \in \mathbb{R}_+^3 : N_r \leq \frac{\Lambda_h}{\mu_r}\right\}$. Then, Ω is positively invariant and attracting with respect to the flow described by system (1). The rate of change of the humans and rodent populations is given in equation below, it follows that

$$\frac{dN_h(t)}{dt} \leq \Lambda_h - \mu_h N_h(t)$$
$$\frac{dN_r(t)}{dt} \leq \Lambda_r - \mu_r N_r(t)$$

A standard comparison theorem [5] can then be used to show that $N_h(t) \leq N_h(0)e^{-\mu_h t} + \frac{\Lambda_h}{\mu_h}\left(1-e^{-\mu_h t}\right)$ and $N_r(t) \leq N_r(0)e^{-\mu_r t} + \frac{\Lambda_r}{\mu_r}\left(1-e^{-\mu_r t}\right)$. In particular, $N_h(t) \leq \frac{\Lambda_h}{\mu_h}$ and $N_r(t) \leq \frac{\Lambda_r}{\mu_r}$ if $N_h(0) \leq \frac{\Lambda_h}{\mu_h}$ and $N_r(0) \leq \frac{\Lambda_r}{\mu_r}$ respectively.

Thus, the region Ω is positively-invariant. Hence, the system is biologically meaningful and mathematically well-posed in the domain Ω. In this domain it is therefore sufficient to consider the dynamics of the flow generated by the model system (1).

3.3 Stability of the disease-free equilibrium (DFE)

The Lassa fever model (1) has a DFE, obtained by setting the right-hand sides of the equations in the model to zero, given by

$$\xi_0 = \left(S_h^*, E_h^*, I_h^*, R_h^*, S_r^*, E_r^*, I_r^*\right) = \left(\frac{\Lambda_h}{\eta\left(1-e^{-rt}\right)+\mu_h}, 0, 0, 0, \frac{\Lambda_r}{\mu_r}, 0, 0\right).$$

The linear stability of ξ_0 can be established using the next generation operator method [5] on the system (1), the matrices F and V, for the new infection terms and the remaining transfer terms, are, respectively, given by

$$F = \begin{pmatrix} 0 & \beta_2 \varepsilon S_h^* & 0 & \beta_1 \sigma S_h^* \\ 0 & 0 & 0 & 0 \\ 0 & \beta_3 \vartheta S_r^* & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } V = \begin{pmatrix} (\kappa + \mu_h) & 0 & 0 & 0 \\ -\kappa & (\phi + \mu_h + \delta) & 0 & 0 \\ 0 & 0 & (\alpha + \mu_r) & 0 \\ 0 & 0 & -\alpha & \mu_r \end{pmatrix}$$

It follows that the reproduction number of the Lassa fever system (1), denoted by R_0, is

$$R_0 = \frac{1}{2}\left(\frac{\beta_2 \varepsilon \kappa \Lambda_h}{(\kappa+\mu_h)(\phi+\mu_h+\delta)\left(\eta(1-e^{-rt})+\mu_h\right)} + \sqrt{\left(\frac{\beta_2 \varepsilon \kappa \Lambda_h}{(\kappa+\mu_h)(\phi+\mu_h+\delta)\left(\eta(1-e^{-rt})+\mu_h\right)}\right)^2 + \frac{4\alpha\sigma\vartheta\kappa\beta_1\beta_3\Lambda_h\Lambda_r}{\mu_r^2(\alpha+\mu_r)(\kappa+\mu_h)(\phi+\mu_h+\delta)\left(\eta(1-e^{-rt})+\mu_h\right)}} \right)$$

The result below follows Theorem 2 in [5].

Lemma 1: The DFE (ξ_0) of the model (1) is locally asymptomatically stable (LAS) if $R_0 < 1$, and unstable if $R_0 > 1$.

Epidemiologically, this implies that Lassa fever will be eliminated from the population whenever $R_0 < 1$ if the initial size of the sub-populations are in the basin of attraction of the DFE i.e. a small influx of Lassa fever infectious individuals into the community will not generate a large Lassa fever outbreak and the disease dies out in time.

3.4 Existence of endemic equilibrium point (EEP)

Next conditions for the existence of endemic equilibria for the model (1) is explored. Let

$$\xi_1 = \left(S_h^*, E_h^*, I_h^*, R_h^*, S_r^*, E_r^*, I_r^*\right),$$

be the arbitrary endemic equilibrium of model (1), in which at least one of the infected components of the model is non-zero.

Let

$$\lambda_h^* = \beta_1 \sigma I_r + \beta_2 \varepsilon I_h + \eta\left(1 - e^{-rt}\right)$$
$$\lambda_r^* = \beta_3 \vartheta I_h$$

be the force of infection in humans and in the vector. Setting the right-hand sides of the equations in (1) to zero gives the following expressions (in terms of λ_h^* and λ_r^*)

$$S_h^* = \left[\frac{(\kappa+\mu_h)(\phi+\delta+\mu_h)}{\kappa\lambda_h^*}\right]\frac{(\psi+\mu_h)\kappa\lambda_h^*\Lambda_h}{(\psi+\mu_h)(\lambda_h^*+\mu_h)(\kappa+\mu_h)(\phi+\delta+\mu_h)-\kappa\lambda_h^*\phi(1-\nu)(\psi+\mu_h)-\kappa\lambda_h^*\psi\phi\nu}$$

$$E_h^* = \left[\frac{(\phi+\delta+\mu_h)}{\kappa}\right]\frac{(\psi+\mu_h)\kappa\lambda_h^*\Lambda_h}{(\psi+\mu_h)(\lambda_h^*+\mu_h)(\kappa+\mu_h)(\phi+\delta+\mu_h)-\kappa\lambda_h^*\phi(1-\nu)(\psi+\mu_h)-\kappa\lambda_h^*\psi\phi\nu}$$

$$R_h^* = \left[\frac{\phi\nu}{(\psi+\mu_h)}\right]\frac{(\psi+\mu_h)\kappa\lambda_h^*\Lambda_h}{(\psi+\mu_h)(\lambda_h^*+\mu_h)(\kappa+\mu_h)(\phi+\delta+\mu_h)-\kappa\lambda_h^*\phi(1-\nu)(\psi+\mu_h)-\kappa\lambda_h^*\psi\phi\nu}$$

$$I_h^* = \frac{(\psi+\mu_h)\kappa\lambda_h^*\Lambda_h}{(\psi+\mu_h)(\lambda_h^*+\mu_h)(\kappa+\mu_h)(\phi+\delta+\mu_h)-\kappa\lambda_h^*\phi(1-\nu)(\psi+\mu_h)-\kappa\lambda_h^*\psi\phi\nu}$$

$$S_r^* = \frac{\Lambda_r}{\lambda_r^*+\mu_r}, E_r^* = \frac{\lambda_r^*\Lambda_r}{(\alpha+\mu_r)(\lambda_r^*+\mu_r)}, I_r^* = \frac{\alpha\lambda_r^*\Lambda_r}{\mu_r(\alpha+\mu_r)(\lambda_r^*+\mu_r)}$$

Substituting into the force of infection in humans and in the vector, gives $a_0\lambda_h^{*2}+b_0\lambda^*+c_0=0$, where

$$a_0 = \mu_r(\alpha+\mu_r)(\psi+\mu_h)(\kappa+\mu_h)(\phi+\delta+\mu_h)\left[\beta_3\vartheta\kappa\mu_r\Lambda_r(\psi+\mu_h)+(\psi+\mu_h)(\kappa+\mu_h)(\phi+\delta+\mu_h)\right]$$

$$b_0 = \mu_r\mu_h(\alpha+\mu_r)(\psi+\mu_h)(\kappa+\mu_h)(\phi+\delta+\mu_h)\left[\beta_3\vartheta\kappa\mu_r\Lambda_r(\psi+\mu_h)+(\psi+\mu_h)(\kappa+\mu_h)(\phi+\delta+\mu_h)\right]$$
$$+\mu_r(\alpha+\mu_r)\left((\psi+\mu_h)(\kappa+\mu_h)(\phi+\delta+\mu_h)\right)^2-\frac{(\psi+\mu_h)^2}{\mu_r\mu_h}(\kappa+\mu_h)(\phi+\mu_h+\delta)\left(\eta(1-e^{-rt})+\mu_h\right)\left(\mu_r^2(\alpha+\mu_r)\right)^2R_0^2$$
$$+\frac{(\psi+\mu_h)^2}{\mu_r\mu_h}(\kappa+\mu_h)(\phi+\mu_h+\delta)\left(\eta(1-e^{-rt})+\mu_h\right)\left(\mu_r^2(\alpha+\mu_r)\right)^2$$

$$c_0 = \mu_r\mu_h(\alpha+\mu_r)(\psi+\mu_h)(\kappa+\mu_h)(\phi+\delta+\mu_h)\left[(\psi+\mu_h)(\kappa+\mu_h)(\phi+\delta+\mu_h)-\beta_2\varepsilon\kappa(\psi+\mu_h)\Lambda_r\right]$$
$$-\beta_1\beta_3\alpha\sigma\vartheta\kappa\mu_r\mu_h\Lambda_r^2(\psi+\mu_h)(\psi+\mu_h)(\kappa+\mu_h)(\phi+\delta+\mu_h)$$

To establish the existence of the endemic equilibrium point, we use the following theorem.

Theorem 3: The Lassa fever model with (special case, $\eta=\phi=0$) has one unique EEP if $b_0<0$. So for the EEP to exist, we need $b_0<0$, that is:

$$\mu_r\mu_h(\alpha+\mu_r)(\psi+\mu_h)(\kappa+\mu_h)(\delta+\mu_h)\left[\beta_3\vartheta\kappa\mu_r\Lambda_r(\psi+\mu_h)+(\psi+\mu_h)(\kappa+\mu_h)(\delta+\mu_h)\right]$$
$$+\mu_r(\alpha+\mu_r)\left((\psi+\mu_h)(\kappa+\mu_h)(\delta+\mu_h)\right)^2-\frac{(\psi+\mu_h)^2}{\mu_r}(\kappa+\mu_h)(\mu_h+\delta)\left(\mu_r^2(\alpha+\mu_r)\right)^2R_0^2$$
$$+\frac{(\psi+\mu_h)^2}{\mu_r}(\kappa+\mu_h)(\mu_h+\delta)\left(\mu_r^2(\alpha+\mu_r)\right)^2<0$$

which implies

$$\mu_r\mu_h(\alpha+\mu_r)(\psi+\mu_h)(\kappa+\mu_h)(\delta+\mu_h)\left[\beta_3\vartheta\kappa\mu_r\Lambda_r(\psi+\mu_h)+(\psi+\mu_h)(\kappa+\mu_h)(\delta+\mu_h)\right]$$
$$+\mu_r(\alpha+\mu_r)\left((\psi+\mu_h)(\kappa+\mu_h)(\delta+\mu_h)\right)^2(1-R_0^2)<0 \Rightarrow R_0>1$$

By this result, then the theorem below gives a condition for the existence of the EEP with $\eta=\phi=0$.

Theorem 4: The EEP with $\eta = \phi = 0$ exists if and only if $R_0 > 1$.

3.5 Global stability analysis for the DFE

Theorem 5: If $R_0 < 1$, the DFE point of the model system (1) is globally asymptotically stable and unstable if $R_0 > 1$.

Proof:

By the comparison theorem, the rate of change of the variables representing the infected components of model system (1) can be re-written as:

$$\begin{bmatrix} E_h^{'} \\ I_h^{'} \\ E_r^{'} \\ I_r^{'} \end{bmatrix} = (F - V)\begin{bmatrix} E_h \\ I_h \\ E_r \\ I_r \end{bmatrix} - \begin{bmatrix} \eta S_h^*\left(1-e^{-rt}\right) \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where the matrices $F\ and\ V$ are defined above. But we also note that $S_h^* \leq \dfrac{\Lambda}{\eta S_h^*\left(1-e^{-rt}\right)+\mu_h}$, $\forall t \geq 0 \in \Omega$. Thus $\begin{bmatrix} E_h^{'} \\ I_h^{'} \\ E_r^{'} \\ I_r^{'} \end{bmatrix} \leq (F - V)\begin{bmatrix} E_h \\ I_h \\ E_r \\ I_r \end{bmatrix}$

using the fact that the eigenvalues of the matrix $(F - V)$ all have negative real parts, it follows that the linearized differential inequality system above is stable whenever $R_0 < 1$ [5]. Consequently, using the system of equations (1), $(E_h, I_h, E_r, I_r) \Rightarrow (0,0,0,0)$ as $t \to \infty$, and evaluating system (1) at $E_h = I_h = E_r = I_r = 0$ gives $S_h^* \to \dfrac{\Lambda}{\eta\left(1-e^{-rt}\right)+\mu_h}$, for $R_0 < 1$. Hence, the DFE point is globally asymptotically stable for $R_0 < 1$.

3.6 Global stability of endemic equilibrium for a special case

In this section, we investigate the global stability of the endemic equilibrium of the model (1), for the special case when $\psi = 0, v = 1$, that there is no loss of immunity.

Theorem 6: The unique endemic equilibrium, ε_1, of the model (1) is GAS whenever $R_{|\psi=0,v=1} > 1$.

Proof:

Let $R_0 > 1$, so that the unique endemic equilibrium, ε_1, exists. Consider the nonlinear Lyapunov function of the Goh-Volterra type

$$F = S_h^{**}\left(\frac{S_h}{S_h^{**}} - \text{In}\frac{S_h}{S_h^{**}}\right) + E_h^{**}\left(\frac{E_h}{E_h^{**}} - \text{In}\frac{E_h}{E_h^{**}}\right) + \frac{\kappa+\mu_h}{\kappa} I_h^{**}\left(\frac{I_h}{I_h^{**}} - \text{In}\frac{I_h}{I_h^{**}}\right) + \frac{(\phi+\delta+\mu_h)(\kappa+\mu_h)}{\kappa\phi} I_h^{**}\left(\frac{R_h}{R_h^{**}} - \text{In}\frac{R_h}{R_h^{**}}\right)$$
$$+ S_r^{**}\left(\frac{S_r}{S_r^{**}} - \text{In}\frac{S_r}{S_r^{**}}\right) + E_r^{**}\left(\frac{E_r}{E_r^{**}} - \text{In}\frac{E_r}{E_r^{**}}\right) + \frac{\alpha+\mu_r}{\alpha} I_r^{**}\left(\frac{I_r}{I_r^{**}} - \text{In}\frac{I_r}{I_r^{**}}\right)$$

The Lyapunov derivative is

$$\dot{F} = \left(1 - \frac{S_h^{**}}{S_h}\right)\dot{S}_h + \left(1 - \frac{E_h^{**}}{E_h}\right)\dot{E}_h + \frac{\kappa+\mu_h}{\kappa}\left(1 - \frac{I_h^{**}}{I_h}\right)\dot{I}_h + \frac{(\phi+\delta+\mu_h)(\kappa+\mu_h)}{\kappa\phi}\left(1 - \frac{R_h^{**}}{R_h}\right)\dot{R}_h$$
$$+ \left(1 - \frac{S_r^{**}}{S_r}\right)\dot{S}_r + \left(1 - \frac{E_r^{**}}{E_r}\right)\dot{E}_r + \frac{\alpha+\mu_r}{\alpha}\left(1 - \frac{I_r^{**}}{I_r}\right)\dot{I}_r$$

Substituting the expressions for the derivatives in $\dot{F}$ (from (1) with $\psi = 0, v = 1$) gives

$$\dot{F} = \Lambda_h - \lambda_h S_h - \mu_h S_h - \frac{S_h^{**}}{S_h}\left(\Lambda_h - \lambda_h S_h - \mu_h S_h\right) + \lambda_h S_h - (\kappa+\mu_h) - \frac{E_h^{**}}{E_h}\left(\lambda_h S_h - (\kappa+\mu_h)E_h\right) +$$
$$\frac{\kappa+\mu_h}{\kappa}\left(\kappa E_h - (\phi+\delta+\mu_h)I_h\right) - \frac{\kappa+\mu_h}{\kappa}\frac{I_h^{**}}{I_h}\left(\kappa E_h - (\phi+\delta+\mu_h)I_h\right) + \frac{(\phi+\delta+\mu_h)(\kappa+\mu_h)}{\kappa\phi}\left(\phi I_h - \mu_h R_h\right) - \frac{(\phi+\delta+\mu_h)(\kappa+\mu_h)}{\kappa\phi}\frac{R_h^{**}}{R_h}\left(\phi I_h - \mu_h R_h\right)$$
$$+ \Lambda_r - \lambda_r S_r - \mu_r S_r - \frac{S_r^{**}}{S_r}\left(\Lambda_r - \lambda_r S_r - \mu_r S_r\right) + \lambda_r S_r - (\alpha+\mu_r)E_r + \frac{E_r^{**}}{E_r}\left(\lambda_r S_r - (\alpha+\mu_r)E_r\right) + \frac{\alpha+\mu_r}{\alpha}\frac{I_r^{**}}{I_r}\left(\alpha E_r - \mu_r I_r\right)$$

so that

$$\dot{F} = \lambda_h S_h^{**}\left(1 - \frac{S_h^{**}}{S_h}\right) + \mu_h S_h^{**}\left(2 - \frac{S_h}{S_h^{**}} - \frac{S_h^{**}}{S_h}\right) + \lambda_h S_h^{**} - \frac{E_h^{**}}{E_h}\lambda_h S_h + (\kappa+\mu_h)E_h^{**} - (\kappa+\mu_h)\frac{I_h^{**}}{I_h}E_h + \frac{(\phi+\delta+\mu_h)(\kappa+\mu_h)}{\kappa}I_h^{**} - \frac{(\phi+\delta+\mu_h)(\kappa+\mu_h)}{\kappa}\frac{R_h^{**}}{R_h}I_h$$
$$+ \frac{(\phi+\delta+\mu_h)(\kappa+\mu_h)\mu_h}{\kappa\phi}R_h - \frac{(\phi+\delta+\mu_h)(\kappa+\mu_h)\mu_h}{\kappa\phi}R_h + \lambda_r S_r^{**}\left(1 - \frac{S_r^{**}}{S_r}\right) + \mu_r S_r^{**}\left(2 - \frac{S_r}{S_r^{**}} - \frac{S_r^{**}}{S_r}\right) + \lambda_r S_r^{**} - \frac{E_r^{**}}{E_r}\lambda_r S_r + (\alpha+\mu_r)E_r^{**} - (\alpha+\mu_r)\frac{I_r^{**}}{I_r}E_r$$
$$+ \frac{(\alpha+\mu_r)\mu_r}{\alpha}I_r^{**} - \frac{(\alpha+\mu_r)\mu_r}{\alpha}I_r$$

Finally, the equation can be further simplified to give

$$\dot{F} = \mu_h S_h^{**}\left(2 - \frac{S_h^{**}}{S_h} - \frac{S_h}{S_h^{**}}\right) + (\kappa+\mu_h)E_h^{**}\left(5 - \frac{S_h^{**}}{S_h} - \frac{E_h^{**}}{E_h} - \frac{E_h}{E_h^{**}}\frac{I_h^{**}}{I_h} - \frac{I_h}{I_h^{**}}\frac{R_h^{**}}{R_h} - \frac{R_h}{R_h^{**}}\right) + \mu_r S_r^{**}\left(2 - \frac{S_r^{**}}{S_r} - \frac{S_r}{S_r^{**}}\right) + (\alpha+\mu_r)E_r^{**}\left(4 - \frac{S_r^{**}}{S_r} - \frac{E_r^{**}}{E_r} - \frac{E_r}{E_r^{**}}\frac{I_r^{**}}{I_r} - \frac{I_r}{I_r^{**}}\right)$$

Since the arithmetic mean exceeds the geometric mean, it follows that

$$2 - \frac{S_h^{**}}{S_h} - \frac{S_h}{S_h^{**}} \le 0, 2 - \frac{S_r^{**}}{S_r} - \frac{S_r}{S_r^{**}} \le 0, 4 - \frac{S_r^{**}}{S_r} - \frac{E_r^{**}}{E_r} - \frac{E_r}{E_r^{**}}\frac{I_r^{**}}{I_r} - \frac{I_r}{I_r^{**}} \le 0, 5 - \frac{S_h^{**}}{S_h} - \frac{E_h^{**}}{E_h} - \frac{E_h}{E_h^{**}}\frac{I_h^{**}}{I_h} - \frac{I_h}{I_h^{**}}\frac{R_h^{**}}{R_h} - \frac{R_h}{R_h^{**}} \le 0$$

Since all the model parameters are non-negative, it follows that $\dot{F} \le 0$ for $R_{|\psi=0,v=1} > 1$. Thus, it follows from the LaSalle's Invariance Principle, that every solution to the equations in the model (1) approaches the EEP, ε_1, as $t \to \infty$ whenever $R_{|\psi=0,v=1} > 1$.

Conclusion

In this paper, we presented a Lassa fever model using a deterministic system of differential equations. The model was qualitatively analyzed for the existence and stability of the disease-free equilibrium, ξ_0, and endemic equilibrium, ξ_1, points. The disease-free equilibrium has been shown to be both locally asymptotically stable and globally asymptotically stable, the later being shown using comparison theorem. The basic reproduction number, $R_0 < 1$, which is an important parameter in the control of Lassa fever infection, was calculated using the next generation method. We have also shown that the endemic equilibrium point, exists for $R_0 > 1$ and has been noted that this endemic equilibrium is unique and globally asymptotically stable based on Lyapunov Function. The implication of the above result to public health is that if the control strategy for Lassa fever can force the reproduction number to below the value of one, then Lassa fever can be eliminated from the population. This model will be extended in future papers to consider incorporation of isolation programme, quarantine, personal production treatment, condom use, trap/cat use, education campaign and rodenticides for Lassa fever epidemic.

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