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On the nilpotents of the semigroup of partial contractions of a finite chain

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Abstract

Let $[n] = \{1, 2, \ldots, n\}$ denote a finite chain. Consider the semigroups $\mathcal{P}_n$ and $\mathcal{OI}_n$, consisting of all partial, and all injective and order-preserving transformations on $[n]$, respectively. Furthermore, let $\mathcal{CP}_n = \{\alpha \in \mathcal{P}_n : |x\alpha - y\alpha| \leq |x - y| \ \ \forall\, x, y \in \text{ Dom } \alpha\}$ and $\mathcal{OCI}_n = \{\alpha \in \mathcal{OI}_n : |x\alpha - y\alpha| \leq |x - y| \ \ \forall\, x, y \in \text{ Dom } \alpha\}$. It follows that $\mathcal{CP}_n$ and $\mathcal{OCI}_n$ are subsemigroups of $\mathcal{P}_n$ and $\mathcal{OI}_n$, respectively. In this paper, we characterize the nilpotent elements in $\mathcal{CP}_n$, compute the number of nilpotents of height $(n-1)$ in $\mathcal{CP}_n$, and determine the number of nilpotents in $\mathcal{OCI}_n$ of a specific type.

Keywords and phrases: Transformations semigroup; Contraction maps; order preserving; nilpotent.

1 Introduction

Let $[n] = \{1, 2, \ldots, n\}$ denote a finite chain. A transformation of $[n]$ is considered *partial* if its domain is a subset of $[n]$ and *full* (or *total*) if its domain is $[n]$. When $\alpha \in S$ (where S is any transformation semigroup), we denote Dom α, Im α, $h(\alpha)$, and $b(\alpha)$ as the domain, image, height (denoted as $|\text{Im }\alpha|$), and width or breadth (denoted as $|\text{Dom }\alpha|$) of α, respectively. A transformation α is described as *order preserving* (or *order reversing*) if, for all $x, y \in \text{Dom }\alpha$, $x \leq y$ implies $x\alpha \leq y\alpha$ (or $x\alpha \geq y\alpha$). Denote $\mathcal{P}_n$, $\mathcal{T}_n$, and $\mathcal{OI}_n$ as the semigroups of partial, full, and one-one order-preserving transformations of $[n]$, respectively. A map $\alpha \in \mathcal{P}_n$ (resp., $\alpha \in \mathcal{T}_n$) is referred to as a *contraction* if $|x\alpha - y\alpha| \leq |x - y|$ for all x, y in Dom α (resp., x, y in $[n]$). Let

$$\mathcal{CP}_n = \{\alpha \in \mathcal{P}_n : |x\alpha - y\alpha| \leq |x - y| \ \ \forall\, x, y \in \text{Dom }\alpha\} \tag{1}$$

and

$$\mathcal{OCI}_n = \{\alpha \in \mathcal{OI}_n : |x\alpha - y\alpha| \leq |x - y| \ \ \forall\, x, y \in \text{Dom }\alpha\}. \tag{2}$$

It is well known that $\mathcal{CP}_n$ and $\mathcal{OCI}_n$ are subsemigroups of $\mathcal{P}_n$ and $\mathcal{OI}_n$, respectively. Denote

$$J_r = \{\alpha \in S : \ h(\alpha) = r\},$$

meaning that J_r contains elements of S with a height of r. For $\alpha, \beta \in S$, the composition of α and β is defined as $\alpha \circ \beta(x) = ((x)\alpha)\beta$ for any x in Dom α. Without ambiguity, we shall use the notation $\alpha\beta$ to denote $\alpha \circ \beta$. When $A = \{a_1, a_2, \ldots, a_n\} \subseteq [n]$ such that $a_1 < a_2 < \ldots < a_n$, the gap(A) is defined as

$$\text{gap}(A) = (|a_2 - a_1|, |a_3 - a_2|, \ldots, |a_n - a_{n-1}|)\,,$$

which yields an $(n-1)$-tuple of numbers. Similarly, the *reverse* gap of A is defined as

$$\text{gap}(A)^R = (|a_n - a_{n-1}|, |a_{n-1} - a_{n-2}|, \ldots, |a_3 - a_2|, |a_2 - a_1|) \ \ (\text{see [1]}).$$

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For a foundational understanding of semigroup theory, readers are encouraged to reference the works of Preston, Mazorchuk, and Howie. These are available in [2, 3, 4].

The algebraic study of the semigroup of contractions has attracted the interest of researchers recently. For example, Zho and Yang [5] characterized the regularity and Green's equivalences for the semigroup of order-preserving partial contractions, denoted as $\mathcal{OCP}_n$. In 2017, Garba et al. [6] investigated the algebraic properties of a certain semigroup of full contractions, providing a characterization of Green's and starred Green's equivalences of the semigroup $\mathcal{CT}_n$ (where $\mathcal{CT}_n$ denotes the semigroup of full contractions on $[n]$).

We also note that certain algebraic properties of some semigroups of contractions have been recorded recently. The concept of semigroups of contractions on chains has been studied by many authors in recent times, as seen in [5, 6, 7, 8, 9]. Combinatorial questions were recently explored by Zubairu and Ali [8] for the semigroups of partial and full contractions on chains. This paper attempts to address some of the questions raised in [8] for the semigroup of partial contractions.

Let S be a semigroup with a zero element. An element $a \in S$ is said to be a *nilpotent* if there exists a positive integer n such that $a^n = 0$. Gomes and Howie [10] demonstrated that $\mathcal{I}_n \setminus \mathcal{S}_n := \mathcal{SI}_n$ is nilpotent-generated if n is even, and they also characterized the nilpotent-generated subsemigroup for odd n. A similar result was obtained by Sullivan [11] for the semigroup $\mathcal{P}_n \setminus \mathcal{T}_n := \mathcal{SP}_n$. Furthermore, Laradji and Umar [12] characterized the nilpotent elements in $\mathcal{P}_n$ and computed the order of nilpotents in the semigroups $\mathcal{P}_n$ and $\mathcal{I}_n$. However, it appears that such characterizations for the semigroup of partial contractions is yet to be investigated, and the number of nilpotents in $\mathcal{CP}_n$ remains an open problem. Nevertheless, in this paper, we provide a characterization of nilpotents in $\mathcal{CP}_n$ and further compute the number of nilpotents of a specific type in $\mathcal{OCI}_n$.

We will now briefly describe the structure of the paper. Section one discusses some basic concepts and the background of the studies. In section two, we provide a certain characterization of nilpotents in $\mathcal{CP}_n$, which serves as a foundation for achieving some of the results in the subsequent sections. In section three, we compute the number of nilpotents in $\mathcal{CP}_n$ whose height is $(n-1)$. Finally, in section 4, we compute the number of nilpotents in $\mathcal{OCI}_n$ of a certain type.

2 Characterizations of nilpotent elements

In this section, we aim to characterize the nilpotent elements in $\mathcal{CP}_n$. Before we commence our investigation, we introduce the following lemma, attributed to Laradji and Umar [12].

Lemma 2.1 ([12]). *An element α in $\mathcal{P}_n$ is nilpotent if and only if there is no nonempty subset A of Dom (α) such that $A\alpha = A$.*

Remark 2.2. Notice that Lemma 2.1 implies that an element $\alpha \in \mathcal{P}_n$ is nilpotent if and only if it has no fixed point and does not form a *cycle*. A cycle occurs when there exist $x, y \in \text{Dom}\,\alpha$ such that $x\alpha = y$ and $y\alpha = x$. In other words, if α is of the form

$$\alpha = \begin{pmatrix} d_1 & d_2 & \dots & d_r & d_{r+1} & d_{r+2} \\ a_1 & a_2 & \dots & a_r & d_{r+2} & d_{r+1} \end{pmatrix},$$

then α has a cycle, represented by

$$\begin{pmatrix} d_{r+1} & d_{r+2} \\ d_{r+2} & d_{r+1} \end{pmatrix},$$

where $d_{r+1}\alpha = d_{r+2}$ and $d_{r+2}\alpha = d_{r+1}$.

Now, let $\alpha = \begin{pmatrix} d_1 & d_2 & \dots & d_r \\ a_1 & a_2 & \dots & a_r \end{pmatrix} \in \mathcal{CP}_n$, where $d_1 \leq d_2 \leq \dots \leq d_r$. We define the smallest element in the domain of α as $\min \text{Dom}\,(\alpha) = d_1$, referred to as the *first* element, and the largest element in the domain as $\max \text{Dom}\,(\alpha) = d_r$, denoted as the *last* element.

For the purpose of illustrations, consider $\alpha = \begin{pmatrix} 1 & 3 & 5 & 6 \\ 3 & 4 & 6 & 7 \end{pmatrix} \in \mathcal{CP}_7$. Then clearly 1 is the first element in the domain of α and 6 is the last element in the domain of α.

Next, let $\alpha \in \mathcal{CP}_n$ then Dom $\alpha \subseteq [n]$ and it is partitioned by the relation $\ker \alpha = \{(x, y) \in \text{Dom } \alpha \times \text{Dom } \alpha : x\alpha = y\alpha\}$ into r blocks A_i $(1 \leq i \leq r)$. As such Dom $\alpha = A_1 \cup \cdots \cup A_r$ where $r \leq n$. As such every $\alpha \in \mathcal{CP}_n$ can be expressed as:

$$\begin{pmatrix} A_1 & A_2 & \ldots & A_r \\ a_1 & a_2 & \ldots & a_r \end{pmatrix},$$

where $r \leq n$ and $|a_i - a_j| \leq |x - y|$ for all $x \in A_i$ and $y \in A_j$ $(i, j \in \{1, 2, \ldots, r\})$.

Let $N(\mathcal{CP}_n)$ be the set of all nilpotents in $\mathcal{CP}_n$. Then we have:

Lemma 2.3. *For $n \geq 2$, $N(\mathcal{CP}_n)$ is not a semigroup.*

Proof. Take $\alpha = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\beta = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \in N(\mathcal{CP}_n)$. Then $\alpha\beta = \begin{pmatrix} 1 \\ 2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin N(\mathcal{CP}_n)$.

We now have the following lemma that characterizes nilpotent elements in $\mathcal{CP}_n$.

Lemma 2.4. *If α is a nilpotent in $\mathcal{CP}_n$, and i and j $(1 \leq i < j \leq n)$ are the first and last elements of the domain of α, then $\{i, j\} \nsubseteq Im(\alpha)$.*

Proof. Suppose, by way of contradiction, that $\alpha \in N(\mathcal{CP}_n)$ is a contraction with a domain containing i and j as the first and last elements, where $1 \leq i < j \leq n$. Assuming further, by way of contradiction, that $\{i, j\} \subseteq \text{Im}(\alpha)$ implies the existence of x and y in Dom (α) such that $x\alpha = i$ and $y\alpha = j$. However, as α is a nilpotent, Lemma 2.1 implies that $\{i, j\} = \{x\alpha, y\alpha\} = \{x, y\}\alpha \neq \{x, y\} \subseteq \text{Dom } (\alpha)$, i.e., $\{i, j\} \neq \{x, y\}$. We then consider three cases:

i $x = i$ and $y < j$,

ii $y = j$ and $x > i$, or

iii $i < x < j$ and $i < y < j$.

We shall demonstrate a contradiction for all the cases.

i Suppose $x = i$ and $y < j$ then $y\alpha - x\alpha = j - i > y - i = y - x$ which implies that $|y\alpha - x\alpha| > |y - x|$. This contradicts the fact that α is a contraction.

ii Suppose that $y = j$ and $x > i$ then $y\alpha - x\alpha = j - i > j - x = y - x$. Notice that $y\alpha - x\alpha$ and $y - x$ are positive, thus we have $|y\alpha - x\alpha| > |y - x|$, which contradicts the fact that α is a contraction.

iii Suppose that $i < x < j$ and $i < y < j$ then $y\alpha - x\alpha = j - i >= y - x$. Notice that $y\alpha - x\alpha$ and $y - x$ are positive, as such we have $|y\alpha - x\alpha| > |y - x|$. This also contradicts the fact that α is a contraction.

Hence $\{i, j\} \nsubseteq \text{Im } \alpha$.

For the purpose of illustrations we give the following example:

Example 2.5. Consider $\alpha = \begin{pmatrix} 1 & 3 & 5 & 6 \\ 3 & 4 & 6 & 7 \end{pmatrix} \in \mathcal{CP}_7$. Notice that 6 is the last element in Dom α and $6 \in \text{Im } \alpha$. However, it is easy to see that, replacing any element in Im $\alpha \setminus \{6\}$ with the first element 1, will render the new α not to be a contraction.

3 On the number of nilpotents in $\mathcal{CP}_n$

In this section, we aim to compute the number of nilpotents in $\mathcal{CP}_n$ with a height of $(n-1)$. Before commencing our investigation, we will first establish the following preliminaries. We denote J_{n-1} as the set of all elements in $\mathcal{CP}_n$ with a height of $(n-1)$, and $N(J_{n-1})$ denotes the set of all nilpotents in J_{n-1}. A subset B of $[n]$ is defined as *convex* if, whenever $x, y \in B$ with $x \leq y$ and $x \leq z \leq y$ for any $z \in [n]$, then $z \in B$. We will now present the following lemma describing the nilpotents in J_{n-1} with a convex domain.

Lemma 3.1. *Let $\alpha \in \mathcal{CP}_n$ and let $\alpha \in J_{n-1}$ be such that the domain of α is convex, of the form $\{1, \ldots, n-1\}$ or $\{2, \ldots, n\}$. Then $\alpha \in N(J_{n-1})$ if and only if α is equal to $\begin{pmatrix} 1 & 2 & \ldots & n-1 \\ 2 & 3 & \ldots & n \end{pmatrix}$ or α is equal to $\begin{pmatrix} 2 & 3 & \ldots & n \\ 1 & 2 & \ldots & n-1 \end{pmatrix}$.*

Proof. Let Dom $(\alpha) = \{1, \ldots, n-1\}$ or $\{2, \ldots, n\}$, where $\alpha \in J_{n-1}$. Suppose α is a nilpotent. Therefore, because Dom (α) is convex, it implies that the image of α must also be convex. It is noteworthy that the sets $\{2, 3, \ldots, n\}$ and $\{1, 2, \ldots, n-1\}$ are the only convex subsets of $[n]$ of order $n-1$. If we take Dom $(\alpha) = \{1, \ldots, n-1\}$, then all the possible contractions with the image $\{2, 3, \ldots, n\}$ or $\{1, 2, \ldots, n-1\}$ are:

$$\begin{pmatrix} 1 & 2 & \ldots & n-3 & n-2 & n-1 \\ 2 & 3 & \ldots & n-2 & n-1 & n \end{pmatrix}, \begin{pmatrix} 1 & 2 & \ldots & n-3 & n-2 & n-1 \\ n & n-1 & \ldots & 4 & 3 & 2 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 & \ldots & n-3 & n-2 & n-1 \\ 1 & 2 & \ldots & n-3 & n-2 & n-1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 2 & \ldots & n-3 & n-2 & n-1 \\ n-1 & n-2 & \ldots & 3 & 2 & 1 \end{pmatrix}.$$

Notice that by Lemma 2.1, the first element $\begin{pmatrix} 1 & 2 & \ldots & n-3 & n-2 & n-1 \\ 2 & 3 & \ldots & n-2 & n-1 & n \end{pmatrix}$ is a nilpotent. However, by Lemma 2.1 the element $\begin{pmatrix} 1 & 2 & \ldots & n-3 & n-2 & n-1 \\ n & n-1 & \ldots & 4 & 3 & 2 \end{pmatrix}$ is not a nilpotent since it has a cycle $\begin{pmatrix} 2 & n-1 \\ n-1 & 2 \end{pmatrix}$.

Similarly, the elements

$$\begin{pmatrix} 1 & 2 & \ldots & n-3 & n-2 & n-1 \\ 1 & 2 & \ldots & n-3 & n-2 & n-1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 & \ldots & n-3 & n-2 & n-1 \\ n-1 & n-2 & \ldots & 3 & 2 & 1 \end{pmatrix}$$

are not nilpotents since the first is a partial identity with $(n-1)$ fixed points and the second element has a cycle $\begin{pmatrix} 1 & n-1 \\ n-1 & 1 \end{pmatrix}$.

This shows that the only nilpotent $\alpha \in J_{n-1}$ with domain of the form $\{1, \ldots, n-1\}$ and image of the form $\{2, 3, \ldots, n\}$ or $\{1, 2, \ldots, n-1\}$ is

$$\begin{pmatrix} 1 & 2 & \ldots & n-3 & n-2 & n-1 \\ 2 & 3 & \ldots & n-2 & n-1 & n \end{pmatrix}.$$

One can show using similar argument that the only nilpotent in J_{n-1} with domain of the form $\{2, \ldots, n\}$ and image of the form $\{2, 3, \ldots, n\}$ or $\{1, 2, \ldots, n-1\}$ is

$$\begin{pmatrix} 2 & 3 & \ldots & n-2 & n-1 & n \\ 1 & 2 & \ldots & n-3 & n-2 & n-1 \end{pmatrix}.$$

Conversely, suppose $\alpha = \begin{pmatrix} 1 & 2 & \dots & n-1 \\ 2 & 3 & \dots & n \end{pmatrix}$ or $\alpha = \begin{pmatrix} 2 & 3 & \dots & n \\ 1 & 2 & \dots & n-1 \end{pmatrix}$. Then it is easy to see that by Lemma 2.1, both elements are nilpotents, each of height $n-1$.

Lemma 3.2. *Let $\alpha \in \mathcal{CP}_n$ ($n \geq 5$) be such that $\alpha \in J_{n-1}$. Suppose Dom $\alpha = \{1, \dots, n-2, n\}$ or $\{1, 3, \dots, n-1, n\}$ then α is not a nilpotent.*

Proof. Suppose Dom $(\alpha) = \{1, \dots, n-2, n\}$, and and suppose by way of contradiction that α is a nilpotent. Since the height of α is $n-1$ and the domain of α is a union of $\{1, \dots, n-2\}$ and $\{n\}$ (i.e., a union of two convex subsets of $[n]$), we observe from Lemma 2.4 that 1 and n cannot simultaneously belong to Im(α). This implies that Im(α) can be either $\{1, 2, \dots, n-2, n-1\}$ or $\{2, \dots, n-1, n\}$. Taking Im$(\alpha) = \{1, 2, \dots, n-2, n-1\}$ yields:

$$\alpha = \begin{pmatrix} 1 & 2 & \dots & n-3 & n-2 & n \\ 1 & 2 & \dots & n-3 & n-2 & n-1 \end{pmatrix}$$

which has at least one fixed point and, according to Lemma 2.1, is not a nilpotent. Alternatively, taking Im$(\alpha) = \{1, 2, \dots, n-2, n-1\}$ results in:

$$\alpha = \begin{pmatrix} 1 & 2 & \dots & n-3 & n-2 & n \\ n-1 & n-2 & \dots & 3 & 2 & 1 \end{pmatrix}$$

which includes a cycle $\begin{pmatrix} 2 & n-2 \\ n-2 & 2 \end{pmatrix}$ and, by Lemma 2.1, is not a nilpotent. This contradiction holds in each case, establishing the result. A similar argument demonstrates that α is not a nilpotent if Dom $(\alpha) = \{1, 3, \dots, n\}$. This completes the proof.

Lemma 3.3. *Let $\alpha \in J_{n-1}$ ($n \geq 5$) be such that Dom $(\alpha) = \{1, 2, \dots, i, i+2, i+3, \dots, n\}$ with gap(Dom $(\alpha)) = (1, 1, \dots, 1, 2, 1, \cdots, 1)$ (an $(n-1)$-tuple of numbers) where 2 occupies the k^{th} position in the tuple ($2 \leq k \leq n-2$). Then, α is nilpotent if and only if $k = i$ (for n odd) or $k = i$ or $i+1$ (for n even) and Im $(\alpha) = \{2, \dots, n\}$ or $\{1, \dots, n-1\}$.*

Proof. Suppose, by way of contradiction, that α is a nilpotent and $k \neq i$ (for n odd) and $k \neq i$ and $k \neq i-1$ (for n even); or Im $(\alpha) \neq \{2, \dots, n\}$ and Im $(\alpha) \neq \{1, \dots, n-1\}$. This implies that if n is odd, then k can take any value, such as $i-l$ or $i+l$ for any $l \geq 1$. As α is a contraction, it falls into one of the following forms:

$$\alpha = \begin{pmatrix} 1 & 2 & \dots & i-1 & i & i+2 & i+3 & \dots & n \\ n & n-1 & \dots & i-l+1 & i-l & i-l-1 & i-l-2 & \dots & 2 \end{pmatrix}$$

or

$$\alpha = \begin{pmatrix} 1 & 2 & \dots & i-1 & i & i+2 & i+3 & \dots & n \\ n & n-1 & \dots & i+l & i+l-1 & i+l-2 & i+l-3 & \dots & 2 \end{pmatrix}$$

Notice that for all $1 \leq l \leq k$, the two maps above are contractions. Furthermore, if $l = 1$, the first map will exhibit the property $i-1 \mapsto i$ and $i \mapsto i-1$, demonstrating a cycle, while the second map will have $i \mapsto i$, signifying i as a fixed point, indicating that these maps are not nilpotent. This contradicts the assumption that they are nilpotent. A similar contradiction arises for even n.

If Im $(\alpha) \neq \{1, \dots, n-1\}$ and Im $(\alpha) \neq \{2, \dots, n\}$, then observe that the height of α is $n-1$. Therefore, if Im (α) is not convex, either $\{1, 2, \dots, i\}\alpha$ is not convex or $\{i+2, \dots, n\}\alpha$ is not convex, contradicting the fact that α is a contraction.

Conversely, suppose $k = i$ (for n odd) or $k = i$ or $i-1$ (for n even) and let Im $(\alpha) = \{2, \dots, n\}$ or Im $(\alpha) = \{1, \dots, n-1\}$.

For odd n, as α is a contraction and Dom (α) is the union of two convex sets (Dom $(\alpha) = \{1, \dots, n\} \cup \{i+2, \dots, n\}$), then (Dom $(\alpha))\alpha = (\{1, \dots, n\})\alpha \cup (\{i+2, \dots, n\})\alpha$ is a union of two convex sets. Hence,

$$\alpha = \begin{pmatrix} 1 & 2 & \dots & i-1 & i & i+2 & i+3 & \dots & n \\ n & n-1 & \dots & i+3 & i+2 & i+1 & i & \dots & 2 \end{pmatrix}$$

or

$$\alpha = \begin{pmatrix} 1 & 2 & \dots & i-1 & i & i+2 & i+3 & \dots & n \\ n-1 & n-2 & \dots & i+2 & i+1 & i & i-1 & \dots & 1 \end{pmatrix}$$

It is evident by Lemma 2.1 that both maps are nilpotent.

For even n,

$$\alpha = \begin{pmatrix} 1 & 2 & \dots & i-1 & i & i+2 & i+3 & \dots & n \\ n-1 & n & \dots & i+2 & i+1 & i & i-1 & \dots & 1 \end{pmatrix}$$

or

$$\alpha = \begin{pmatrix} 1 & 2 & \dots & i-1 & i & i+2 & i+3 & \dots & n \\ n & n-1 & \dots & i+2 & i+1 & i & i-1 & \dots & 2 \end{pmatrix}.$$

It easily follows from Lemma 2.1 that the two maps are nilpotents. This completes the proof.

We now complete the section with the main result.

Theorem 3.4. *Let $\mathcal{CP}_n$ $(n \geq 5)$ be as defined in (1), then $|N(J_{n-1})| = 4$.*

Proof. Notice that by Lemma 3.1, Lemma 3.2, and Lemma 3.3, the domain of a nilpotent α in J_{n-1} is of the form $\{1, \dots, n-1\}$ or $\{2, \dots, n\}$ or Dom $(\alpha) = \{1, 2, \dots, i, i+2, i+3, \dots, n\}$ with gap(Dom $(\alpha)) = (1, 1, \cdots, 1, 2, 1, \cdots, 1)$ (an $n-1$ tuple of numbers) with 2 occupying the k^{th} position in the tuple $(2 \leq k \leq n-2)$. Thus, by Lemma 3.1, each of the domains $\{1, \dots, n-1\}$ or $\{2, \dots, n\}$ gives rise to only one nilpotent, and by Lemma 3.3, the domain Dom $(\alpha) = \{1, 2, \dots, i, i+2, i+3, \dots, n\}$ with gap (Dom $(\alpha)) = (1, 1, \cdots, 1, 2, 1, \cdots, 1)$ gives rise to only two nilpotents. This completes the proof.

4 On the number of nilpotents in $\mathcal{OCI}_n$

In this section we compute the number of nilpotents in $\mathcal{OCI}_n$ whose domain is $\{1, 2, \dots, i\}$ and gap(Dom $\alpha) = (1, 1, \dots, 1, k)$ of height equal $r = i - k + 1$, where $k \leq i \leq n$.

Let $S = \mathcal{IO}_n$ be the semigroup of one-to-one order-preserving partial transformations of $[n]$. The following lemma, attributed to Garba [13], characterizes nilpotent elements in $\mathcal{IO}_n$.

Lemma 4.1. *Let $\alpha \in \mathcal{IO}_n$ be such that $h(\alpha) < n$, then α is a nilpotent if and only if $x\alpha \neq x$ for every $x \in$ Dom α.*

Remark 4.2. Notice that since every nilpotent element of $\mathcal{OCI}_n$ is also a nilpotent element of $\mathcal{IO}_n$, then the above Lemma also holds in $\mathcal{OCI}_n$.

Let $P_r([n])$ be the collection of all subsets of $[n]$, each of order equal to r. For illustrative purposes, consider the cases when $n = 3$, i.e., $[3] = \{1, 2, 3\}$ and $r = 2$, we see that

$$P_2([3]) = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\};$$

and when $[n] = \{1, 2, 3, 4\}$ and $r = 3$,

$$P_3([4]) = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}.$$

Following the pattern in Section 1.2 of [14], we present the following lemma:

Lemma 4.3. *If $[n] = \{1, 2, \dots, n\}$ then $|P_r([n])| = \binom{n}{r}$.*

Corollary 4.4 (Problem 1.39, [14]). $\sum_{r=0}^{n}|P_r([n])| = 2^n$.

Recall that if $A = \{a_1, a_2, \ldots, a_n\} \subseteq [n]$ such that $a_1 < a_2 < \ldots < a_n$, then gap (A) is defined as

$$\text{gap}(A) = (|a_2 - a_1|, |a_3 - a_2|, \ldots, |a_n - a_{n-1}|),$$

which represents an $(n-1)$-tuple of numbers. Similarly, the reverse gap of A is defined as gap $(A)^R = (|a_n - a_{n-1}|, |a_{n-1} - a_{n-2}|, \ldots, |a_3 - a_2|, |a_2 - a_1|)$.

For example, if $A = \{1, 2, 4, 5, 8\}$, then gap $(A) = (1, 2, 1, 3)$ and $gap(A)^R = (3, 1, 2, 1)$.

Remark 4.5. (gap $(A)^R)^R =$ gap (A).

Define a relation $\sim$ on $P_r([n])$ as follows: for all A, B in $P_r([n])$, $A \sim B$ if and only if gap $(A) =$ gap (B) or gap $(A) =$ gap $(B)^R$. It is easy to see that the relation $\sim$ is an equivalence on $P_r([n])$.

Let $[A]$ denote the equivalence class of A under the relation $\sim$, then we have the following lemma:

Lemma 4.6. *Every equivalence class contains an element that includes 1*

Proof. Let $A \in P_r([n])$ and suppose $[A]$ is the equivalence class of A under the relation $\sim$. Now let B be an element in $[A]$, which means that $B \sim A$, i.e., gap$(B) =$ gap(A) or gap$(B) =$ gap$(A)^R$. Without loss of generality, suppose gap$(B) =$ gap(A). This means that

$$(|b_2 - b_1|, |b_3 - b_2|, \ldots, |b_n - b_{n-1}|) = (|a_2 - a_1|, |a_3 - a_2|, \ldots, |a_n - a_{n-1}|),$$

where $a_i \in A$ and $b_i \in B$ for $1 \leq i \leq r$, which implies that $|b_{i+1} - b_i| = |a_{i+1} - a_i|$ for $1 \leq i \leq r$. Now suppose $|a_{i+1} - a_i| = k_i$ for $1 \leq i \leq r$, this implies that $|b_{i+1} - b_i| = k_i$ for $1 \leq i \leq r$. It is clear that $k_i \leq n-1$ for any i, which means that $k_i + 1 \leq n$ for any i, implying that $k_i + 1 \in [n]$ for any i. As such, choosing $b_{i+1} = k_i + 1$ then $b_i = 1$ for any i, implies $1 \in B$. Similarly, choosing gap$(B) =$ gap$(A)^R$ gives the same result. This completes the proof.

Lemma 4.7. $|[A]| = n - k - r + 2$, *where* $|A| = r$ *and gap* $(A) = (1, 1, \ldots, 1, k)$ *or* $gap(A)^R = (1, 1, \ldots, 1, k)$.

Proof. Let A be an element of $P_r([n])$, where $A = \{a_1, a_2, \ldots, a_{r-1}, a_r\}$ such that $|a_{i+1} - a_i| = 1$ for $1 \leq i \leq r-2$ and $|a_r - a_{r-1}| = k$. Now, by Lemma 2.4, there exists a B in the equivalence class of A which contains 1. Thus, $B = \{1, 2, 3, \ldots, r-1, r-1+k\}$. Then, by the definition of the relation $\sim$ and the fact that $k \leq n-1$, we have:

$$[B] = \{\{1, 2, 3, \ldots, r-1, r-1+k\}, \{2, 3, \ldots, r, r+k\}, \{3, 4, \ldots, r+1, r+1+k\}, \ldots, \{n-k-r+2, n-k-r+3, \ldots, n\}\},$$

which implies that $|[B]| = n - k - r + 2$.

Now as a consequence of the above lemma, we already have the following (now) obvious result.

Corollary 4.8. *For any natural number* n *there are* $n - k - r + 2$ *possible subsets of* $[n]$ *of order* r *with gap of the form* $(1, 1, \ldots, 1, k)$, *which is an* $(r-1)$*-tuple of number(s).*

Lemma 4.9. *Let* $[i - k + 1] = \{1, 2, 3, \ldots, i-k-1, i-k, i\}$, *then the number of nilpotents in* $\mathcal{OCI}_n$ *whose domain is* $[i-k+1]$ *is* $\frac{(k-2)(k-1)}{2}$.

Proof. Let $[i-k+1] = \{1, 2, 3, \ldots, i-k-1, i-k, i\}$. Then $|[i-k+1]| = i-k+1$ and $k = |i - (i-k)|$, now let Dom $\alpha = [i-k+1]$, then by Lemma 2.4, 1 and i cannot be in the image set of α, and notice that there are $k-1$ jumps (by jumps here we are referring to missing elements between $i-k$ and i) between $i-k$ and i, then the image of α is of the form $\{2, 3, \ldots, i-k, i-k+1, i-k+2, \ldots, k-2, k-1\} = [k] \setminus \{1, k\}$, which gives a total of $k-2$ elements.

Now we can define a one to one order preserving partial contraction map from Dom α to any subset of $\{2,3,\ldots,k-1\}$ of order i in $\binom{k-i}{k-i-1}$ ways $(2 \leq i \leq k-1)$, and summing over i, we have:

$$\begin{aligned}\sum_{i=2}^{k-1}\binom{k-i}{k-i-1} &= \binom{k-2}{k-3}+\binom{k-3}{k-4}+\ldots+\binom{k-(k-2)}{k-(k-2)-1}+\binom{k-(k-1)}{k-(k-1)-1}\\ &= \binom{k-2}{k-3}+\binom{k-3}{k-4}+\ldots+\binom{4}{3}+\binom{3}{2}+\binom{2}{1}+\binom{1}{0}\\ &= (k-2)+(k-3)+\cdots+4+3+2+1\\ &= \frac{(k-2)(k-1)}{2}.\end{aligned}$$

Corollary 4.10. *Let $S = \mathcal{OCI}_n$, then the number of nilpotents α in S whose domain is $\{1,2,3,\ldots,i-k,i\}$ and image a subset of $\{2,3,\ldots,k-1\}$ is $\frac{(k-2)(k-1)}{2}$.*

Lemma 4.11. *Let $[n] = \{1,2,\ldots,i-k-1,i-k,i-k+1,i-k+2,\ldots,i-k+k,i+1,i+2,\ldots,n\}$ and Dom $\alpha = \{1,2,\ldots,i-k-1,i-k,i\}$. Then the number of one to one order preserving partial contractions of height equal to $i-k+1$, whose image is a subset of $\{i-k+1,i-k+2,\ldots,i,i+1,i+2,\ldots,n\}$ (i.e., a set of order $n+k-r$) is $k(n-i)$.*

Proof. Notice that since 1 is in the domain of α, then 1 has the following possibilities: $1 \to i-k+1$ or $1 \to i-k+2$ or $\ldots$ up to $1 \to i$ within the set $\{i-k+1,i-k+2,\ldots,i,i+1,i+2,\ldots,n\}$ of order $n+k-r$. This can be accomplished in a total of $i-(i-k+1)+1$ ways, which simplifies to k ways (i.e., since the last term is $i-k+k=i$).

In a similar fashion, we can do the same to the remaining elements in the domain in $n-k-m+1$ ways, where $m = h(\alpha) = i-k+1$, which is $(n-k-(i-k+1)+1) = (n-i)$ ways, hence applying product rule, we have a total of $k(n-i)$ ways.

Lemma 4.12. *Let $S = \mathcal{OCI}_n$. The number of nilpotents α in S whose domain is $\{1,2,\ldots,i\}$, gap (Dom α) $=$ $(1,1,\ldots,k)$ and $h(\alpha) = i-k+1$ is $(n-k-i+2)\left(\frac{(k-2)(k-1)}{2}+k(n-i)\right) = (n-r+3)\left(\frac{(k-2)(k-1)}{2}+k(n-r-k+1)\right)$ where $r = i-k+1$ and $1 \leq k \leq n-1$.*

Proof. The proof follows directly from Lemma 4.7, Corollary 4.10 and Lemma 4.11.

Now as a consequence of the above Lemma4.12 we now conclude this section with the main result.

Theorem 4.13. *Let $\mathcal{OCI}_n$ be as expressed in (2). Then the number of nilpotents in $\mathcal{OCI}_n$ whose domain is $\{1,2,\ldots,i\}$ and gap (Dom α) $= (1,1,\ldots,1,k)$ of height equal $r = i-k+1$, is $\sum\limits_{k=1}^{n-1}(n-r+3)\left(\frac{(k-2)(k-1)}{2}+k(n-r-k+1)\right)$.*

Remark 4.14. The number of nilpotent elements in the semigroups $\mathcal{CP}_n$ and $\mathcal{OCI}_n$ has not been investigated.

5 Conclusion

We have successfully given a characterization of nilpotents in the semigroup of partial contractions of a finite chain. We also computed the order of nilpotents of height $n-1$ in $\mathcal{CP}_n$ and computed the order of certain one to one order preserving nilpotents contractions.

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