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Approximate common solution of variational inequality and fixed point problems for family of nonexpansive semigroup

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Abstract

In this paper, we prove a path convergence theorem and introduce a new iterative scheme for approximation of common element of fixed point sets of family of nonexpansive semigroup and the set of solutions of variational inequality problem for an α-inverse strongly accretive mapping in strictly convex q-uniformly smooth real Banach space. As an application, solution is given to the problem of finding a common element of fixed point sets of family of nonexpansive semigroup and the set of zeros of an α-inverse strongly accretive operator. The result obtained augment and generalize several existing results.

Keywords and phrases: Modulus of smoothness; Modulus of convexity; generalized duality maps; α-inverse strongly accretive operators; k-strictly pseudocontractive mappings.

1 Introduction

Let E be a real normed linear space. The modulus of convexity of E is the function $\delta_E : [0,2] \to [0,1]$ defined by

$$\delta_E(\epsilon) = inf\Big\{1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1,\ \epsilon = \|x-y\|\Big\}.$$

The space E is said to be uniformly convex if and only if $\delta_E(\epsilon) > 0\ \forall\ \epsilon \in (0,2]$. E is said to be *strictly convex* if for all $x, y \in E$ such that $\|x\| = \|y\| = 1$ and for all $\lambda \in (0,1)$ we have $\|\lambda x + (1-\lambda)y\| < 1$. It is well known that every uniformly convex Banach space is strictly convex.

Let $S := \{x \in E : \|x\| = 1\}$. The space E is said to have a *Gâteaux differentiable* norm if the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each $x, y \in S$. When this limit exists, we say that E is smooth. E is said to have a *uniformly Gâteaux differentiable* norm if for each $y \in S$ the limit is attained uniformly for $x \in S$. Furthermore, E is said to be uniformly smooth if the limit exists uniformly for $(x,y) \in S \times S$. It is well known that every uniformly smooth real Banach space is a reflexive real Banach space (see e.g., [[13]]).

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Let E be a real normed linear space with dimension, $dim(E) \geq 2$. The *modulus of smoothness* of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(t) = \sup\Big\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = t\Big\}.$$

In terms of the modulus of smoothness, the space E is called uniformly smooth if and only if $\lim_{t\to 0^+} \frac{\rho_E(t)}{t} = 0$. E is called *q-uniformly smooth* if there exists a constant $c > 0$ such that $\rho_E(t) \leq ct^q,\ t > 0$. It is easy to see that for $1 < q < +\infty$, every q-uniformly smooth Banach space is uniformly smooth; and thus has a uniformly Gâteaux differentiable norm.

Let E be a real normed linear space with dual E^*. We denote by J_q *the generalized duality mapping* from E to 2^{E^*} defined by

$$J_q x := \{f^* \in E^* : \langle x, f^*\rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1}\},$$

where $\langle ., .\rangle$ denotes the generalized duality pairing between members of E and members of E^*. For $q = 2$, the mapping $J = J_2$ of E to 2^{E^*} is called *the normalized duality mapping*. It is well known that if E is uniformly smooth or E^* is strictly convex, then duality mapping is single-valued; and if E has a uniformly Gâteaux differentiable norm then the duality mapping is norm-to-weak* uniformly continuous on bounded subsets of E. If $E = H$ is a Hilbert space then the duality mapping becomes the identity map of H (see e.g., [[50]]). In the sequel, we shall denote the single-valued generalized duality mapping by j_q and the single valued normalized duality mapping by j.

Let K be a closed convex nonempty subset of a real normed space. One parameter family $\Psi := \{T(t) : t \in \mathbb{R}^+\}$, where $\mathbb{R}^+$ denotes the set of nonnegative real numbers, is said to be a (continuous) Lipschitzian semigroup (see e.g. [[47]]) of mappings from K to K if the following conditions are satisfied:

1. $T(0)x = x \ \forall\ x \in K$;
2. $T(s+t)x = T(s)T(t)x \ \forall\ s, t \in \mathbb{R}^+,\ x \in K$;
3. for each $t > 0$, there exists a bounded measurable function $L_t : [0, +\infty) \to [0, +\infty)$ such that $\|T(t)x - T(t)y\| \leq L_t\|x - y\| \ \forall\ x, y \in K$;
4. for each $x \in K$, the mapping $T(.)x$ from $\mathbb{R}^+$ into K is continuous.

A Lipschitzian semigroup is called nonexpansive if $L_t = 1$ for all $t > 0$ and asymptotically nonexpansive if $\limsup_{t\to\infty} L_t \leq 1$. We shall denote by
$F(\Psi)$ the set of common fixed points of the semigroup Ψ, that is

$$F(\Psi) = \{x \in K : T(t)x = x,\ \forall\ t \in \mathbb{R}^+\} = \bigcap_{t\in\mathbb{R}^+} F(T(t)).$$

As a motivation for study of semigroup of operators, consider the Cauchy problem

$$\begin{cases} u'(t) = Au(t), t \geq 0, \\ u(0) = x, \end{cases} \tag{1}$$

where $A = (a_{ij})$ is an $n \times n$ matrix with $a_{ij} \in \mathbb{C}$ for $i, j = 1, 2, ..., n$, and x is a given vector in $\mathbb{C}^n$. It is well-known that the above Cauchy problem has a unique solution given by

$$u(t) = e^t Ax,\ t \geq 0,$$

where e^tA represents the fundamental matrix of the linear differential system $u'(t) = Au(t)$; and equals I for $t = 0$. We have

$$e^tA = \sum_{k=1}^{\infty} \frac{t^k}{k!} A_k,$$

valid for all $t \in \mathbb{R}$. Here, A and e^tA can be interpreted as linear operators, $A \in L(X)$, $e^tA \in L(X)$, where $X = \mathbb{C}^n$, equipped with any of its equivalent norms.

Note that the family of matrices (operators) $\{T(t) = e^tA, t \geq 0\}$ is a (uniformly continuous) semi-group on $X = \mathbb{C}^n$. Even more, $\{T(t) : t \geq 0\}$ extends to a group of linear operators, $\{e^tA : t \in \mathbb{R}\}$. The representation of the solution $u(t)$ as

$$u(t) = T(t)x, t \geq 0, \tag{2}$$

allows the derivation of some properties of the solution from the properties of the family $\{T(t) : t \geq 0\}$. This idea extends easily to the case in which X is a general Banach space and A is a bounded linear operator, $A \in L(X)$.

If A is a linear unbounded operator, $A : D(A) \subseteq X \to X$, with some additional conditions, then one can associate with A a so-called C_0-semigroup of linear opera- tors $\{T(t) \in L(X) : t \geq 0\}$, such that the solution of the Cauchy problem is again represented by the above formula (2). In this way one can solve various linear Partial Diferencial Equations (PDEs), where A represents linear unbounded differential operator with respect to the spacial variables, defined on convenient function spaces.It is worth mentioning that the connection between A and the corresponding semigroup $\{T(t) : t \geq 0\} \subset L(X)$ is established by the well-known Hille-Yosida theorem.

Linear semigroup theory received considerable attention in the 1930s as a new approach in the study of linear parabolic and hyperbolic partial differential equations. Note that the linear semigroup theory has later developed as an independent theory, with applications in some other fields, such as ergodic theory, the theory of Markov processes, and so on. It is well known that importance of the system (1) in modeling the dynamism of evolution system in Mathematical Biology cannot be over emphasized.

Iterative approximation of fixed points and zeros of nonlinear operators have been studied extensively by many authors to solve nonlinear operator equations as well as variational inequality problems (see e.g., [1], [6], [7], [8], [14], [15], [18], [25]-[29], [12], [20], [21], [31], [43], [32]-[36], [47], [51]). Construction of fixed points of nonexpansive semigroups is an important subject in nonlinear operator theory and its applications, in particular, in image recovery and signal processing (see e.g., [9, 30, 49]). Most published results on family of nonexpansive semigroup centered on the iterative approximation of common fixed points of the family. Shioji and Takahashi (1998) introduced the following implicit iteration process for approximation of common fixed point of family of nonexpansive semigroup

$$x_n = \alpha_n u + (1 - \alpha_n)\sigma_{t_n}(x_n),\ n \geq 1, \tag{3}$$

where $\{\alpha_n\}_{n\geq 1}$ is a sequence in $(0, 1)$ and $\{t_n\}_{n\geq 1}$ is a sequence of positive real numbers such that $\lim_{n\to\infty} t_n = +\infty$; and for $t > 0$, σ_t is given by $\sigma_t(x) = \frac{1}{t}\int_0^1 T(s)xds$. Under suitable conditions on the sequence $\{\alpha_n\}_{n\geq 1}$, Shioji and Takahashi [37] proved the convergence of (3) to an element of $F(\Psi)$.

Chen and Song [11] introduced the following implicit and explicit viscosity iteration processes

$$x_n = \alpha_n f(x_n) + (1 - \alpha_n)\sigma_{t_n} x_n,\ n \geq 1 \tag{4}$$

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)\sigma_{t_n} x_n,\ n \geq 1 \tag{5}$$

and proved that these sequences converge to some element of $F(\Psi)$ in uniformly convex Banach space with uniformly Gâteaux differentiable norm.

Suziki [40] proved the convegence of the iteration scheme

$$x_n = \alpha_n u + (1-\alpha_n)T(t_n)x_n,\ n \geq 1 \tag{6}$$

to a common fixed point of family of nonexpansive semigroup in Hilbert space.

In 2002, Benavides *et al.* [4] proved in uniformly smooth Banach space that if Ψ satisfies some asymptotic regularity condition and $\{\alpha_n\}_{n\geq 1}$ satisfies some mild conditions, then the implicit scheme (6) and the explicit scheme

$$x_{n+1} = \alpha_n u + (1-\alpha_n)T(t_n)x_n,\ n \geq 1 \tag{7}$$

converges strongly to an element of $F(\Psi)$.

In [48], Xu studied the strong convergence of the implicit iteration processes (3) and (6) in uniformly convex Banach space which admits a weakly sequential continuous duality mapping.

Recently, Song and Xu [39], studied the convergence of the following implicit and explicit viscosity iteration processes in strictly convex reflexive real Banach space with uniformly Gâteaux differentiable norm

$$x_n = \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n,\ n \geq 1$$

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n,\ n \geq 1$$

They improved and generalized the corresponding results of Chen and Song [11], Aleyner and Censor [2] and Aleyner and Reich [?], Benavides *et al.* [4].

In this paper, it is our purpose to prove a path convergence theorem and construct a new iterative scheme for approximation of common element of fixed point sets of family of nonexpansive semigroup and the set of solution of variational inequality problem for an α-inverse strongly accretive mapping in q-uniformly smooth real Banach space. As an application, solution is given to the problem of finding a common element of fixed point sets of family of nonexpansive semigroup and the set of zeros of an α-inverse strongly accretive operator. Our iteration schemes, corollaries and method of proof are of independent interest.

2 Methodology/Model formulation

We shall now discuss various tools which we shall make use of in this paper; some of them will be stated as lemmas and propositions.

Let H be a real Hilbert space, a mapping $A : H \to H$ is said to be monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0\ \forall\ x, y \in D(A).$$

The mapping $A : H \to H$ is called α-inverse strongly monotone if there exists a positive real number α such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2\ \forall\ x, y \in D(A).$$

A mapping $T : H \to H$ is said to be k-strictly pseudocontractive if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2 \ \forall\, x, y \in D(T).$$

Observe that if $k = 0$, then the mapping T is nonexpansive. Observe also that if T is a k-strictly pseudocontractive mapping and we put $A = I - T$, where I is the identity operator of H, then we have that

$$\|(I - A)x - (I - A)y\|^2 \leq \|x - y\|^2 + k\|Ax - Ay\|^2 \ \forall\, x, y \in D(A)$$

and since H is a real Hilbert space, we have that

$$\|(I - A)x - (I - A)y\|^2 = \|x - y\|^2 + \|Ax - Ay\|^2 - 2\langle x - y, Ax - Ay\rangle,$$

so that

$$\langle x - y, Ax - Ay\rangle \geq \frac{1 - k}{2}\|Ax - Ay\|^2.$$

Thus, if T is k-strictly pseudocontractive mapping, then $A = I - T$ is α-inverse strongly monotone operator with $\alpha = \frac{1-k}{2}$.

Let K be a closed convex subset of a real Hilbert space H. A variational inequality problem is to find $x^* \in K$ such that

$$\langle Ax^*, y - x^*\rangle \geq 0 \ \forall\, y \in K,$$

where A is a monotone mapping of K to E (see e.g., [22, 24, 42, 29]).

In the sequel, we shall denote the set of solutions of variational inequality problems by $VI(K, A)$.

Let H be a real Hilbert space with inner product $\langle ., .\rangle$ and norm $\|.\|$; and let K be a closed convex subset of H. For any point $u \in H$, there exists a unique $P_K u \in K$ such that

$$\|u - P_K u\| \leq \|u - y\| \ \forall\, y \in K.$$

P_K is called the *metric projection* of H onto K. It is well known that P_K is a nonexpansive mapping of H onto K. It is also known that P_K satisfies

$$\langle x - y, P_K x - P_K y\rangle \geq \|P_K x - P_K y\|^2 \ \forall\, x, y \in H.$$

Furthermore, $P_K x$ is characterized by the properties $P_K x \in K$ and $\langle x - P_K x, P_K x - y\rangle \geq 0 \ \forall\, y \in K$. In the context of variational inequality problem, this implies that

$$u \in VI(K, A) \Leftrightarrow u = P_K(u - \lambda Au) \ \forall\, \lambda > 0.$$

A monotone operator A is said to be *maximal* if the graph, $G(A)$, of A is not properly contained in the graph of any other monotone operator, where $G(A) = \{(x, y) \in H \times H : y = Ax\}$. It is also known that A is maximal monotone if and only if for $(x, f) \in H \times H$, $\langle f - Ay, x - y\rangle \geq 0$ for every $(y, Ay) \in G(A)$ implies $f = Ax$. Let A be an α-inverse strongly monotone mapping defined from K into H and $N_K v$ be a normal cone to K at $v \in K$, that is $N_K v = \{w \in H : \langle w, v - u\rangle \geq 0, \ \forall\, u \in K\}$. Define a map M by

$$Mv = \begin{cases} Av + N_K v, \ v \in K, \\ \emptyset, \ v \notin K. \end{cases} \tag{8}$$

Then, M is maximal monotone and $u \in M^{-1}(0) \ \Leftrightarrow \ u \in VI(K, A)$, (see e.g., [34]).

We now turn to the generalized concept of variational inequality problem. Let E be a real Banach space. A mapping $A : D(A) \subset E \to E$ is said to be *accretive* if for all $x, y \in D(A)$ there exists $j_q(x-y) \in J_q(x-y)$ such that

$$\big\langle Ax - Ay, j_q(x-y)\big\rangle \geq 0$$

The mapping A is called an α-*inverse strongly accretive* if there exists a positive real number α such that $\forall\, x, y \in D(A)$ there exists $j_q(x-y) \in J_q(x-y)$ such that

$$\big\langle Ax - Ay, j_q(x-y)\big\rangle \geq \alpha\|Ax - Ay\|^q.$$

Observe that an α-inverse strongly accretive mapping is $\frac{1}{\alpha^{\frac{1}{q-1}}}$-Lipschitz.

Let K be a closed, convex and nonempty subset of E, Let $A : K \to E$ be an accretive mapping. A variational inequality problem is to find $x^* \in K$ such that for some $j_q(x-x^*) \in J_q(x-x^*)$,

$$\big\langle Ax^*, j_q(x-x^*)\big\rangle \geq 0 \;\forall\, x \in K.$$

We observe that the concept of monotonicity and accretivity coincide if and only if the underlying space is a real Hilbert space.

Let Q be a mapping of E onto K; Q is said to be *sunny* if $Q\big(Qx + t(x - Qx)\big) = Qx$ for all $x \in E$ and $t \geq 0$. A mapping Q of E into E is said to be a retraction if $Q^2 = Q$. If a mapping Q is a retraction, then $Qz = z$ for every $z \in R(Q)$, the range of Q. A subset K is said to be sunny nonexpansive retract of E if there exists a sunny nonexpansive retraction of E onto K. A retraction Q is said to be orthogonal if for each $x \in E,\ x - Q(x)$ is normal to K in the sense of R. C. James [19].

In [10], Bruck established the following:

1. A nonexpansive projection is necessarily an orthogonal retraction; and if E is smooth, then the converse is also true.

2. If E is smooth, then there can exist at most one nonexpansive projection of K onto a given subset C of K.

3. If E is uniformly smooth and there exists a nonexpansive retraction of K onto C, then there exists a nonexpansive projection of K onto C. The proximity mapping is nonexpansive projection in a Hilbert space, but not in a general Banach space.

It is well known (see e.g., [10]) that if E is a smooth real Banach space, then Q is an orthogonal retraction of E onto K if and only if $Q(x) \in K$ and

$$\big\langle Q(x) - x, j_q(Q(x) - y)\big\rangle \leq 0 \;\forall\, y \in K. \tag{9}$$

It then follows that for $x, y \in K$, we have $\big\langle Q(x) - x, j_q(Q(x) - Q(y))\big\rangle \leq 0$ and $\big\langle Q(y) - y, j_q(Q(y) - Q(x))\big\rangle \leq 0$ which implies that

$$\|Q(x) - Q(y)\|^2 \leq \big\langle x - y, j_q(Q(x) - Q(y))\big\rangle. \tag{10}$$

In the context of variational inequality problem, (9) implies that

$$u \in VI(K, A) \Leftrightarrow u = Q(u - \lambda Au) \;\forall\, \lambda > 0,$$

where in this case the mapping $A : K \to E$ is an accretive mapping.

An accretive mapping A is said to be *maximal accretive* if its graph, $G(A)$, is not contained in the graph of any other accretive mapping. Equivalently, A is maximal accretive if for every $(x, f) \in E \times E$ such that $\big\langle f - Ay, j_q(x-y)\big\rangle \geq 0$ holds for all $(y, Ay) \in G(A)$ implies $f = Ax$.

Proposition 2.1. *Let K be a closed convex nonepmty subset of a Banach space E. Let $A : K \to E$ be an α-inverse strongly accretive mapping. Let M be defined by*

$$Mv = \begin{cases} Av + N_K v, \; v \in K, \\ \emptyset, \; v \notin K. \end{cases} \tag{11}$$

where $N_K v = \{w \in E : \big\langle w, j_q(v-u)\big\rangle \geq 0, \;\forall\; u \in K\}$, then M is maximal accretive and $u \in M^{-1}(0) \;\Leftrightarrow\; u \in VI(K, A)$.

Proof. Let $v_0, w_0 \in E$ such that $\big\langle w - w_0, j_q(v - v_0)\big\rangle \geq 0$ holds for all $w \in Mv = Av + N_K v$. Then since $0 \in N_K v$, we have that

$$\big\langle Av - w_0, j_q(v - v_0)\big\rangle \geq 0 \;\forall\; v \in K.$$

Let $w_0 = Av_0 + w_1$, then $\big\langle Av - Av_0, j_q(v - v_0)\big\rangle \geq \big\langle w_1, j_q(v - v_0)\big\rangle$. For $\varepsilon \in (0, 1)$, let $v_\varepsilon = \varepsilon(v - v_0) + v_0$, then by convexity of K we have that $v_\varepsilon \in K$; and so,

$$\big\langle Av_\varepsilon - Av_0, j_q(v_\varepsilon - v_0)\big\rangle \geq \big\langle w_1, j_q(v_\varepsilon - v_0)\big\rangle.$$

Thus, we have the following estimate,

$$\begin{aligned} \varepsilon^{q-1}\big\langle w_1, j_q(v - v_0)\big\rangle &= \big\langle w_1, j_q(v_\varepsilon - v_0)\big\rangle \leq \big\langle Av_\varepsilon - Av_0, j_q(v_\varepsilon - v_0)\big\rangle \\ &\leq \|Av_\varepsilon - Av_0\| \|j_q(\varepsilon(v - v_0))\| \\ &\leq \varepsilon^{q-1} \frac{1}{\alpha^{q-1}} \|v_\varepsilon - v_0\| \|v - v_0\|^{q-1} \\ &= \varepsilon^{q} \frac{1}{\alpha^{q-1}} \|v - v_0\|^{q}. \end{aligned}$$

Thus, $\big\langle w_1, j_q(v - v_0)\big\rangle \leq \varepsilon \frac{1}{\alpha^{q-1}} \|v - v_0\|^q$, so that as $\varepsilon \to 0$, we have

$$\big\langle w_1, j_q(v_0 - v)\big\rangle \geq 0 \;\forall\; v \in K.$$

Hence, $w_1 \in N_K v_0$ and $w_0 \in M v_0$, that is M is maximal accretive. It is therefore obvious that $u \in M^{-1}(0) \;\Leftrightarrow\; u \in VI(K, A)$. □

Let μ be a bounded linear functional defined on ℓ_∞ satisfying $\|\mu\| = 1 = \mu(1)$. It is known that μ is a mean on $\mathbb{N}$ if and only if

$$\inf\{a_n : n \in \mathbb{N}\} \leq \mu(a_n) \leq \sup\{a_n : n \in \mathbb{N}\}$$

for every $a = (a_1, a_2, a_3, ...) \in \ell_\infty$. In the sequel, we shall use $\mu_n(a_n)$ instead of $\mu(a)$. A mean μ on $\mathbb{N}$ is called *a Banach limt* if $\mu_n(a_n) = \mu_n(a_{n+1})$ for every $a = (a_1, a_2, a_3, ...) \in \ell_\infty$. It is well known that if μ is a Banach limit, then

$$\liminf_{n\to\infty} a_n \leq \mu_n a_n \leq \limsup_{n\to\infty} a_n$$

for all $a = (a_1, a_2, a_3, ...) \in \ell_\infty$. Furthermore, if $a = (a_1, a_2, ...)$, $b = (b_1, b_2, ...) \in \ell_\infty$ and $\lim\limits_{n\to\infty} a_n = a^* \left(\lim\limits_{n\to\infty}(a_n - b_n) = 0\right)$, then $\mu_n(a_n) = a^* \Big(\text{repectively, } \mu_n(a_n) = \mu_n(b_n)\Big)$.

In the sequel, we shall make use of the following Lemmas:

Lemma 2.2. *Let E be a real normed linear space, then for $1 < q < +\infty$, the following inequality holds:*

$$\|x + y\|^q \leq \|x\|^q + q\big\langle y, j_q(x + y)\big\rangle \;\forall\; x, y \in E, \;\forall\; j_q(x + y) \in J_q(x + y).$$

In particular, for $q = 2$,

$$\|x + y\|^2 \leq \|x\|^2 + 2\big\langle y, j(x + y)\big\rangle \;\forall\; x, y \in E, \;\forall\; j(x + y) \in J(x + y).$$

Lemma 2.3. *(See e.g., [5, 44, 45]) Let $\{\lambda_n\}_{n\geq 1}$ be a sequence of nonnegative real numbers satisfying the condition*

$$\lambda_{n+1} \leq (1-\alpha_n)\lambda_n + \sigma_n,\ n \geq 0,$$

where $\{\alpha_n\}_{n\geq 0}$ and $\{\sigma_n\}_{n\geq 0}$ are sequences of real numbers such that $\{\alpha_n\}_{n\geq 1} \subset [0,1]$, $\sum_{n=1}^{\infty}\alpha_n = +\infty$. Suppose that $\sigma_n = o(\alpha_n),\ n \geq 0$ (i.e., $\lim_{n\to\infty}\frac{\sigma_n}{\alpha_n} = 0$) or $\sum_{n=1}^{\infty}|\sigma_n| < +\infty$ or $\limsup_{n\to\infty}\frac{\sigma_n}{\alpha_n} \leq 0$, then $\lambda_n \to 0$ as $n \to \infty$.

Lemma 2.4. *(See e.g. [41]) Let $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ be bounded sequences in a real Banach space E and let $\{\gamma_n\}_{n\geq 1}$ be a sequence in $[0,1]$ with $0 < \liminf \gamma_n \leq \limsup_{n\to\infty}\gamma_n < 1$. Suppose that $x_{n+1} = \gamma_n y_n + (1-\gamma_n)x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\|) \leq 0$, then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.*

Lemma 2.5. *(See e.g. [46]) Let E be a q-uniformly smooth real Banach space for some $q > 1$, then there exists some positive constant d_q such that*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + d_q\|y\|^q\ \forall\, x,y \in E,\ \forall\, j_q(x) \in J_q(x). \tag{12}$$

If E is L_q (or ℓ_q) space, the constant d_q in (12) has been calculated. This is shown in the following lemma.

Lemma 2.6. *(See e.g. [23]) Let E be L_q (or ℓ_q) space ($1 < q < +\infty$) and $x, y \in E$,*

1. *if $1 < q < 2$, then*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + d_q\|y\|^q\ \forall\, x,y \in E,\ \forall\, j_q(x) \in J_q(x),$$

where $d_q = \frac{1+b_q^{q-1}}{(1+b_q)^{q-1}}$, and b_q is the unique solution of the equation $(q-2)b^{q-1} + (q-1)b^{q-2} - 1 = 0,\ 0 < b < 1$.

2. *if $2 \leq q < +\infty$, then*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x)\rangle + (q-1)\|y\|^2\ \forall\, x,y \in E,\ \forall\, j(x) \in J(x).$$

Now, let E be a q-uniformly smooth real Banach space. Let $A : K \to E$ be an α-inverse strongly accretive mapping. Then for the map $(I-\lambda A) : K \to E$ (where I is the identity map of K and $\lambda > 0$), we have the following estimates:

$$\begin{aligned}
\|(I-\lambda A)x - (I-\lambda A)y\|^q &= \|x - y - \lambda(Ax - Ay)\|^q \\
&\leq \|x-y\|^q - q\lambda\langle Ax - Ay, j_q(x-y)\rangle \\
&\quad + d_q\lambda^q\|Ax - Ay\|^q \\
&\leq \|x-y\|^q - \lambda(q\alpha - d_q\lambda^{q-1})\|Ax - Ay\|^q.
\end{aligned}$$

If λ is such that $0 < \lambda < \left(\frac{q\alpha}{d_q}\right)^{\frac{1}{q-1}}$, we have that $(I-\lambda A)$ is a nonexpansive mapping of K into E.

For L_q (or ℓ_q) spaces, ($2 \leq q < +\infty$) we have the following:

$$\begin{aligned}
\|(I-\lambda A)x - (I-\lambda A)y\|^2 &= \|x - y - \lambda(Ax - Ay)\|^2 \\
&\leq \|x-y\|^2 - 2\lambda\langle Ax - Ay, j_q(x-y)\rangle \\
&\quad + (q-1)\lambda^2\|Ax - Ay\|^2 \\
&\leq \|x-y\|^2 - \lambda(2\alpha - (q-1)\lambda)\|Ax - Ay\|^q \\
&\leq \|x-y\|^2,
\end{aligned}$$

provided $\lambda \in \left(0, \frac{2\alpha}{q-1}\right)$. In particular, if $E = H$ is a Hilbert space and we choose $\lambda \in (0, 2\alpha)$, then $(I - \lambda A)$ is nonexpansive.

We now present the concept of uniformly asymptotic regular semigroup (see e.g [2]-[4], [39]). Let K be a nonempty closed convex subset of a Banach space E, $\Psi := \{T(t) : t \in \mathbb{R}^+\}$ a continuous operator semigroup on K. Then Ψ is said to be uniformly asymptotic regular on K if for all $h \in \mathbb{R}^+$ and any bounded subset C of K,

$$\lim_{t\to\infty} \sup_{x\in C} \|T(h)(T(t)x) - T(t)x\| = 0.$$

The nonexpansive semigroup $\{\sigma_t : t > 0\}$ defined by the following lemma is an example of uniformly asymptotic regular semigroup of mappings.

Lemma 2.7. *(See [11] Lemma 2.7, [39] lemma 2.2) Let K be a nonempty closed convex subset of a uniformly convex Banach space E, and D a bounded closed convex subset of K, and $\Psi := \{T(t) : t \in \mathbb{R}^+\}$ a nonexpansive semigroup on K such that $F(\Psi) \neq \emptyset$. For each $x \in K$, define $\sigma_t(x) := \frac{1}{t}\int_0^t T(s)xds$, then*

$$\lim_{t\to\infty} \sup_{x\in D} \|T(h)(\sigma_t x) - \sigma_t x\| = 0.$$

Other examples of uniformly asymptotic regular semigroup of mappings can be found in Examples 17 and 18 of [2].

3 Analysis of the model

Proposition 3.1. *Let K be a nonempty closed convex subset of a reflexive strictly convex real Banach space E. Let $\Psi = \{T(t) : t \in \mathbb{R}^+\}$ be a family of nonexpansive semigroup from K into itself. Let $\{x_n\}_{n\geq 1}$, a bounded sequence in K, be an approximate fixed point sequence of Ψ, that is, $\lim_{n\to\infty} \|x_n - T(t)x_n\| = 0 \; \forall \; t \in \mathbb{R}^+$. Let*

$$\Omega := \left\{x \in K \bigcap B : \mu_n\|x_n - x\|^2 = \min_{y\in K} \mu_n\|x_n - y\|^2\right\},$$

for some bounded closed convex nonempty subset, B, of E such that $x_n \in B \; \forall \; n \in \mathbb{N}$. If $F(\Psi) \neq \emptyset$, then $\Omega \bigcap F(\Psi) \neq \emptyset$.

Proof. Fix $x^* \in F(\Psi)$, then since $\{x_n\}_{n\geq 1}$ is bounded, there exists $R > 0$ sufficiently large such that $x_n \in B = \overline{B_R(x^*)}$ for all $n \in \mathbb{N}$. Furthermore, $K \bigcap B$ is a bounded closed convex and nonempty subset of E. Define a map $\phi : E \to \mathbb{R}$ by

$$\phi(y) = \mu_n\|x_n - y\|^2,$$

then ϕ is continuous, convex and coersive (that is $\phi(y) \to \infty$ as $\|y\| \to \infty$). Thus, since E is a reflexive Banach space, there exists $x_0 \in K \bigcap B$ such that

$$\phi(x_0) = \min_{y\in K\bigcap B} \mu_n\|x_n - y\|^2.$$

So, the set

$$\Omega := \left\{x \in K \bigcap B : \mu_n\|x_n - x\|^2 = \min_{y\in K\bigcap B} \mu_n\|x_n - y\|^2\right\} \neq \emptyset.$$

Ω is also bounded, closed and convex. We now show that $\Omega \bigcap F(\Psi) \neq \emptyset$. Let $z \in \Omega$, then since $\lim_{n\to\infty} \|x_n - T(t)x_n\| = 0$ for all $t \in \mathbb{R}^+$, we obtain that for all $t \in \mathbb{R}^+$,

$$\phi\big(T(t)z\big) = \mu_n\|x_n - T(t)z\|^2 \leq \mu_n\|T(t)x_n - T(t)z\|^2 \leq \mu_n\|x_n - z\|^2.$$

This implies that
$$\phi\big(T(t)z\big) \leq \phi(z)\ \forall\ t \in \mathbb{R}^+,$$
so that as $t \to \infty$, using that fact that ϕ is continuous, we obtain that
$$\phi\big(\lim_{t\to\infty} T(t)z\big) \leq \phi(z).$$
Hence, $\lim\limits_{t\to\infty} T(t)z \in \Omega$. Let $p \in F(\Psi)$, then since every closed convex and nonempty subset of a reflexive strictly convex real Banach space E is a Chebyshev set (see e.g. [26], corollary 5.1.19), there exists a uniqe $u^* \in \Omega$ such that $\|p - u^*\| = \inf\limits_{x\in\Omega} \|p - x\|$. Since $\lim\limits_{t\to\infty} T(t)p = p$ and $\lim\limits_{t\to\infty} T(t)u^* \in \Omega$, we have that
$$\|p - \lim_{t\to\infty} T(t)u^*\| = \|\lim_{t\to\infty} T(t)p - \lim_{t\to\infty} T(t)u^*\| = \lim_{t\to\infty} \|T(t)p - T(t)u^*\| \leq \|p - u^*\|.$$
Thus, $\lim\limits_{t\to\infty} T(t)u^* = u^*$. Now, since $T(h+t)x = T(h)T(t)x\ \forall\ h, t \in \mathbb{R}^+,\ x \in \Omega$, then we have that $u^* = \lim\limits_{t\to\infty} T(t)u^* = \lim\limits_{t\to\infty} T(t+h)u^* = \lim\limits_{t\to\infty} T(t)T(h)u^* = T(h) \lim\limits_{t\to\infty} T(t)u^* = T(h)u^*\ \forall\ h \in \mathbb{R}^+$. Hence, $u^* \in F(\Psi)$. This completes the proof. □

Remark 3.2. *In what follows, the real sequences* $\{\alpha_n\}_{n\geq1}$ *and* $\{\sigma_{i,n}\}_{n\geq1},\ i \in \mathbb{N}$ *in* $(0,1)$ *are such that* $\sum\limits_{i=1}^{\infty} \sigma_{i,n} = 1 - \alpha_n$ *and* $\lim\limits_{n\to\infty} \dfrac{\alpha_n}{\sigma_{i,n}} = 0,\ i = 1, 2, ...;$ $\{t_i\}_{i\geq1}$ *is a sequence in* $\mathbb{R}^+$ *such that* $\lim\limits_{i\to\infty} t_i = +\infty$*;* $\delta \in (0,1)$ and $\lambda \in \left(0, \left(\frac{q\alpha}{d_q}\right)^{\frac{1}{q-1}}\right)$ *are fixed.* $\Psi := \{T(t) : t \in \mathbb{R}^+\}$, $FV := F(\Psi) \bigcap VI(K, A) \neq \emptyset$ *and* $F(S) := \{x \in D(S) : Sx = x\}$.

We state without proof the following proposition whose proof is immediate.

Proposition 3.3. *Let K be a nonempty closed and convex nonexpansive retract of a q-uniformly smooth real Banach space E with nonexpansive retraction Q. Let $A : K \to E$ be an α-inverse strongly accretive mapping and Ψ be a family of nonexpansive semigroup from K into itself. Define a map $G_n : K \to K$ by*
$$G_n x = \alpha_n u + (1-\delta)(1-\alpha_n)x + \delta \sum_{i\geq1} \sigma_{i,n} T(t_i) Q(x - \lambda Ax),\ \forall\ x \in K,$$
where $u \in K$ is arbitrary. Then, G_n is a strict contraction and thus, for each $n \in \mathbb{N}$ there exists a unique $z_n \in K$ such that
$$z_n = \alpha_n u + (1-\delta)(1-\alpha_n)z_n + \delta \sum_{i\geq1} \sigma_{i,n} T(t_i) Q(z_n - \lambda Az_n). \tag{13}$$

Lemma 3.4. *Let K be a nonempty closed and convex nonexpansive retract of a q-uniformly smooth real Banach space E with nonexpansive retraction Q. Let $\{z_n\}_{n\geq1}$ be a sequence satisfying (13), then $\{z_n\}$ is bounded.*

Proof. Let $p \in FV$, then using (13), we have
$$\begin{aligned}
\|z_n - p\|^2 &= \big\langle \alpha_n(u-p) + (1-\delta)(1-\alpha_n)(z_n - p) \\
&\quad + \delta \sum_{i\geq1} \sigma_{i,n}\Big(T(t_i)Q(z_n - \lambda Az_n) - p\Big), j(z_n - p)\big\rangle \\
&\leq \alpha_n \big\langle u - p, j(z_n - p)\big\rangle + (1-\alpha_n)(1-\delta)\|z_n - p\|^2 \\
&\quad + \delta \sum_{i\geq1} \sigma_{i,n} \|z_n - p\|^2 \\
&= \alpha_n \big\langle u - p, j(z_n - p)\big\rangle + (1-\alpha_n)\|z_n - p\|^2.
\end{aligned}$$

This implies that
$$\|z_n - p\| \leq \|u - p\| \;\forall\; n \in \mathbb{N}.$$
Hence, $\{z_n\}$ is bounded. $\square$

Lemma 3.5. *Let K be a nonempty closed and convex nonexpansive retract of a q-uniformly smooth real Banach space E with nonexpansive retraction Q such that $q \geq 1+d_q$. Let $\Psi = \{T(t) : t \in \mathbb{R}^+\}$ be a family of uniformly asymptotic regular nonexpansive semigroup from K into itself and let $A : K \to E$ be an α-inverse strongly accretive mapping. Let $\{z_n\}_{n\geq 1}$ be a sequence satisfying (13), then $\lim\limits_{n\to\infty} \|z_n - T(t)z_n\| = 0 \;\forall\; t \in \mathbb{R}^+$.*

Proof. Let $i \in \mathbb{N}$ and $p \in FV$, then using the hypothesis that $q \geq 1 + d_q$, we obtain from Lemma 2.5 that

$$\begin{aligned}
\|T(t_i)Q(z_n - \lambda Az_n) - z_n\|^q &= \|T(t_i)Q(z_n - \lambda Az_n) - p + p - z_n\|^q \\
&\leq \|p - z_n\|^q + q\big\langle T(t_i)Q(z_n - \lambda Az_n) - p, j_q(p - z_n)\big\rangle \\
&\quad + d_q\|T(t_i)Q(z_n - \lambda Az_n) - p\|^q \\
&\leq (1 + d_q)\|z_n - p\|^q \\
&\quad + q\big\langle T(t_i)Q(z_n - \lambda Az_n) - p, j_q(p - z_n)\big\rangle \\
&= (1 + d_q - q)\|z_n - p\|^q \\
&\quad + q\big\langle T(t_i)Q(z_n - \lambda Az_n) - z_n, j_q(p - z_n)\big\rangle \\
&\leq q\big\langle z_n - T(t_i)Q(z_n - \lambda Az_n), j_q(z_n - p)\big\rangle. \qquad (14)
\end{aligned}$$

Using (13), we obtain

$$\begin{aligned}
\big\langle z_n - p, j_q(z_n - p)\big\rangle &= \alpha_n\big\langle u - p, j_q(z_n - p)\big\rangle \\
&\quad + (1 - \alpha_n)(1 - \delta)\big\langle z_n - p, j_q(z_n - p)\big\rangle \\
&\quad + \delta\sum_{i\geq 1}\sigma_{i,n}\big\langle T(t_i)Q(z_n - \lambda Az_n) - z_n + z_n - p, j_q(z_n - p)\big\rangle \\
&= \alpha_n\big\langle u - p, j_q(z_n - p)\big\rangle + (1 - \alpha_n)\big\langle z_n - p, j_q(z_n - p)\big\rangle \\
&\quad + \delta\sum_{i\geq 1}\sigma_{i,n}\big\langle T(t_i)Q(z_n - \lambda Az_n) - z_n, j_q(z_n - p)\big\rangle.
\end{aligned}$$

This implies that
$$\delta\sum_{i\geq 1}\sigma_{i,n}\big\langle z_n - T(t_i)Q(z_n - \lambda Az_n), j_q(z_n - p)\big\rangle = \alpha_n\big\langle u - z_n, j_q(z_n - p)\big\rangle. \qquad (15)$$

Using (14) and (15), we obtain
$$\frac{\delta}{q}\sum_{i\geq 1}\sigma_{i,n}\|T(t_i)Q(z_n - \lambda Az_n) - z_{t_n}\|^q \leq \alpha_n\big\langle u - z_n, j_q(z_n - p)\big\rangle.$$

Thus, since $\sigma_{i,n}$ is positive for each $i,\ n \in \mathbb{N}$ and by Lemma 3.4 $\{z_n\}$ is bounded, we have that

$$\begin{aligned}
\frac{\delta}{q}\sigma_{i,n}\|T(t_i)Q(z_n - \lambda Az_n) - z_n\|^q &\leq \frac{\delta}{q}\sum_{i\geq 1}\sigma_{i,n}\|T(t_i)Q(z_n - \lambda Az_n) - z_n\|^q \\
&\leq \alpha_n\big\langle u - z_n, j_q(z_n - p)\big\rangle \leq \alpha_n R,\ \forall\; i,\ n \in \mathbb{N}
\end{aligned}$$

and for some real constant $R > 0$. So,
$$\|T(t_i)Q(z_n - \lambda Az_n) - z_n\|^q \leq \frac{qR}{\delta}\Big(\frac{\alpha_n}{\sigma_{i,n}}\Big).$$

Hence,

$$\lim_{n\to\infty} \|T(t_i)Q(z_n - \lambda A z_n) - z_n\| = 0 \;\forall\, i \in \mathbb{N}. \tag{16}$$

Since $\{z_n\}_{n\geq 1}$ is bounded, we have that $Q(z_n - \lambda A z_n)$ is also bounded. Thus, for all $t \in \mathbb{R}^+$,

$$\Big\|T(t)\Big(T(t_i)Q(z_n - \lambda A z_n)\Big) - T(t_i)Q(z_n - \lambda A z_n)\Big\| \leq \sup_{x\in C}\Big\|T(t)(T(t_i)x) - T(t_i)x\Big\|,$$

where C is any bounded subset of K containing $Q(z_n - \lambda A z_n)$. Furthermore, since $\lim_{i\to\infty} t_i = +\infty$ and the family Ψ is uniformly asymptotic regular, we have that

$$\lim_{i\to\infty}\sup_{x\in C}\Big\|T(t)(T(t_i)x) - T(t_i)x\Big\| = 0.$$

Thus, for all $t \in \mathbb{R}^+$, we have that

$$\begin{aligned}
\|z_n - T(t)z_n\| &\leq \|z_n - T(t_i)Q(z_n - \lambda A z_n)\| \\
&\quad + \Big\|T(t_i)Q(z_n - \lambda A z_n) - T(t)\Big(T(t_i)Q(z_n - \lambda A z_n)\Big)\Big\| \\
&\quad + \Big\|T(t)\Big(T(t_i)Q(z_n - \lambda A z_n)\Big) - T(t)z_n\Big\| \\
&\leq 2\|z_n - T(t_i)Q(z_n - \lambda A z_n)\| \\
&\quad + \Big\|T(t)\Big(T(t_i)Q(z_n - \lambda A z_n)\Big) - T(t_i)Q(z_n - \lambda A z_n)\Big\| \\
&\leq 2\|z_n - T(t_i)Q(z_n - \lambda A z_n)\| \\
&\quad + \sup_{x\in C}\Big\|T(t)(T(t_i)x) - T(t_i)x\Big\|.
\end{aligned}$$

So,

$$\limsup_{n\to\infty}\|z_n - T(t)z_n\| \leq \sup_{x\in C}\Big\|T(t)(T(t_i)x) - T(t_i)x\Big\|$$

and taking limit as $i \to \infty$ on both sides of this inequality, we obtain

$$\limsup_{n\to\infty}\|z_n - T(t)z_n\| \leq \lim_{i\to\infty}\sup_{x\in C}\Big\|T(t)(T(t_i)x) - T(t_i)x\Big\| = 0.$$

Hence, $\lim_{n\to\infty}\|z_n - T(t)z_n\| = 0 \;\forall\, t > 0$. This completes the proof. □

Theorem 3.6. *Let K be a nonempty closed and convex nonexpansive retract of a strictly convex q-uniformly smooth real Banach space E with nonexpansive retraction Q and such that $q \geq 1 + d_q$. Let Ψ be a family of uniformly asymptotic regular nonexpansive semigroup from K into itself and let $A : K \to E$ be an α-inverse strongly accretive mapping. Let $\{z_n\}$ be a sequence satisfying (13), then $\{z_n\}$ converges strongly to an element of FV.*

Proof. Let $p^* \in FV$, then since the sequence $\{z_n\}_{n\geq 1}$ is bounded, there exists $R > 0$ sufficiently large such that $u, z_n \in B^* := \overline{B_R(p^*)}$ for all $n \in \mathbb{N}$. Define the set

$$K_{min} = \Big\{x \in K \bigcap B^* : \mu_n\|z_n - x\|^2 = \min_{y\in K\bigcap B^*}\|z_n - y\|\Big\}.$$

Then, since $\lim_{n\to\infty}\|z_n - T(t)z_n\| = 0 \;\forall\, t \in \mathbb{R}^+$, we have by Proposition 3.1 that there exists a unique $u^* \in K_{min}$ such that $u^* = T(t)u^* \;\forall\, t \in \mathbb{R}^+$, i.e., $u^* \in F(\Psi)$. Let $\gamma \in (0,1)$. Then, by convexity of $K\bigcap B^*$, we have that $(1-\gamma)u^* + \gamma u \in K\bigcap B^*$. So, $\mu_n\|z_n - u^*\|^2 \leq \mu_n\|z_n - \big((1-\gamma)u^* + \gamma u\big)\|^2$. By Lemma 2.2, we have that

$$\|z_n - u^* - \gamma(u - u^*)\|^2 \leq \|z_n - u^*\|^2 - 2\gamma\big\langle u - u^*, j\big(z_n - u^* - \gamma(u - u^*)\big)\big\rangle.$$

So,

$$\mu_n\|z_n - u^* - \gamma(u-u^*)\|^2 \le \mu_n\|z_n - u^*\|^2 - 2\gamma\mu_n\big\langle u-u^*, j\big(z_n - u^* - \gamma(u-u^*)\big)\big\rangle.$$

This implies that $\mu_n\big\langle u-u^*, j\big(z_n - u^* - \gamma(u-u^*)\big)\big\rangle \le 0$. Furthermore, since the normalized duality mapping j is norm-to-weak* uniformly continuous on bounded subsets of E, we obtain that

$$\lim_{\gamma\to 0}\Big(\big\langle u-u^*, j(z_n-u^*)\big\rangle - \big\langle u-u^*, j\big(z_n - u^* - \gamma(u-u^*)\big)\big\rangle\Big) = 0$$

Thus, for all $\epsilon > 0$, there exists $\eta_\epsilon > 0$ such that for all $\gamma \in (0,\eta_\epsilon)$ and for all $n \in \mathbb{N}$,

$$\big\langle u-u^*, j(z_n-u^*)\big\rangle - \big\langle u-u^*, j\big(z_n - u^* - \gamma(u-u^*)\big)\big\rangle < \epsilon.$$

Therefore, for all $\gamma \in (0,\eta_\epsilon)$,

$$\mu_n\big\langle u-u^*, j(z_n-u^*)\big\rangle - \mu_n\big\langle u-u^*, j\big(z_n - u^* - \gamma(u-u^*)\big)\big\rangle \le \epsilon,$$

so that as $\epsilon \to 0$, we obtain that $\mu_n\big\langle u-u^*, j(z_n-u^*)\big\rangle \le 0$. Using (13), we obtain that

$$\begin{aligned}
\|z_n - u^*\|^2 &= \Big\langle \alpha_n(u-u^*) + (1-\alpha_n)(1-\delta)(z_n-u^*) \\
&\quad +\delta\sum_{i\ge1}\sigma_{i,n}\Big(T(t_i)Q(z_n-\lambda Az_n)-u^*\Big), j(z_n-u^*)\Big\rangle \\
&\le \alpha_n\big\langle u-u^*, j(z_n-u^*)\big\rangle + (1-\alpha_n)\|z_n-u^*\|^2. \qquad (17)
\end{aligned}$$

This implies that

$$\|z_n - u^*\|^2 \le \big\langle u-u^*, j(z_n-u^*)\big\rangle.$$

Thus, $\mu_n\|z_n-u^*\|^2 \le \mu_n\big\langle u-u^*, j(z_n-u^*)\big\rangle \le 0$. So $\mu_n\|z_n-u^*\|^2 = 0$. Thus, there exists a subsequence $\{z_{n_k}\}_{k\ge1}$ of $\{z_n\}_{n\ge1}$ such that $z_{n_k} \to u^*$ as $k\to\infty$.

Claim. $z_n \to u^*$ as $n \to \infty$.

Proof of claim. Suppose $\{z_{n_m}\}_{m\ge1}$ is another subsequence of $\{z_n\}_{n\ge1}$ which converges strongly to q^*. Then, it is clear that $q^* \in F(\Psi)$ since $T(t)$ is continuous and $\lim\limits_{m\to\infty}\|T(t)z_{n_m} - z_{n_m}\| = 0 \;\forall\, t \in \mathbb{R}^+$. Furthermore, since $\lim\limits_{m\to\infty}\|T(t_i)Q(z_{n_m} - \lambda Az_{n_m})\| = 0 \;\forall\, i \in \mathbb{N}$ (see (16)), we have by continuity of $T(t_i)Q(I-\lambda A)$ that $T(t_i)Q(q^*-\lambda Aq^*) = q^* \;\forall\; i \in \mathbb{N}$. We now show that $q^* = u^*$. Suppose for contradiction the $q^* \ne u^*$. Then using (13),

$$\begin{aligned}
\|z_{n_k} - q^*\|^2 &= \Big\langle \alpha_{n_k}(u-q^*) + (1-\delta)(1-\alpha_{n_k})(z_{n_k}-q^*) \\
&\quad +\delta\sum_{i\ge1}\sigma_{i,n_k}\Big(T(t_i)Q(z_{n_k}-\lambda Az_{n_k}\Big)-q^*), j(z_{n_k}-q^*)\Big\rangle \\
&= \alpha_{n_k}\big\langle u-u^*, j(z_{n_k}-q^*)\big\rangle + \alpha_{n_k}\big\langle u^*-z_{n_k}, j(z_{n_k}-q^*)\big\rangle \\
&\quad +\alpha_{n_k}\|z_{n_k}-q^*\|^2 + (1-\delta)(1-\alpha_{n_k})\|z_{n_k}-q^*\|^2 \\
&\quad +\delta\sum_{i\ge1}\sigma_{i,n_k}\Big\langle\Big(T(t_i)Q(z_{n_k}-\lambda Az_{n_k})-q^*\Big), j(z_{n_k}-q^*)\Big\rangle \\
&\le \alpha_{n_k}\big\langle u-u^*, j(z_{n_k}-q^*)\big\rangle + \alpha_{n_k}\big\langle u^*-z_{n_k}, j(z_{n_k}-q^*)\big\rangle \\
&\quad +\alpha_{n_k}\|z_{n_k}-q^*\|^2 + (1-\delta)(1-\alpha_{n_k})\|z_{n_k}-q^*\|^2 \\
&\quad +\delta(1-\alpha_{n_k})\|z_{n_k}-q^*\|^2 \\
&= \alpha_{n_k}\big\langle u-u^*, j(z_{n_k}-q^*)\big\rangle + \alpha_{n_k}\big\langle u^*-z_{n_k}, j(z_{n_k}-q^*)\big\rangle \\
&\quad +\|z_{n_k}-q^*\|^2,
\end{aligned}$$

so that

$$\big\langle u-u^*, j(q^*-z_{n_k})\big\rangle \leq M\|z_{n_k}-u^*\| \tag{18}$$

for some $M>0$. Since the duality mapping j is norm-to-weak* uniformly continuous on bounded subsets of E, we have that

$$\big\langle u-u^*, j(q^*-z_{n_k})\big\rangle \to \big\langle u-u^*, j(q^*-u^*)\big\rangle.$$

Thus, we obtain from (18) that

$$\big\langle u-u^*, j(q^*-u^*)\big\rangle \leq 0. \tag{19}$$

Following the same line of argument, we also obtain that $\big\langle u-q^*, j(u^*-q^*)\big\rangle \leq 0$, which implies that

$$\big\langle q^*-u, j(q^*-u^*)\big\rangle \leq 0. \tag{20}$$

Adding (19) and (20) we obtain that $\big\langle q^*-u^*, j(q^*-u^*)\big\rangle \leq 0$, that is, $\|q^*-u^*\| \leq 0$, a contradiction. So, $q^*=u^*$. Hence, $\{z_n\}_{n\geq 1}$ converges strongly to $u^*\in F(\Psi)$.

Next, we show that $u^*\in VI(K,A)$. Let $x^*\in FV$ and $y_n=Q(z_n-\lambda Az_n)$. Then,

$$\begin{aligned}
\|y_n-x^*\|^q &= \|Q(z_n-\lambda Az_n)-Q(x^*-\lambda Ax^*)\|^q\\
&\leq \|z_n-x^*-\lambda(Az_n-Ax^*)\|^q\\
&\leq \|z_n-x^*\|^q+\lambda(d_q\lambda^{q-1}-q\alpha)\|Az_n-Ax^*\|^q.
\end{aligned}$$

Again, using (13) and the convexity of $\|.\|^q$, we have that

$$\begin{aligned}
\|z_n-x^*\|^q &= \|\alpha_n(u-x^*)+(1-\alpha_n)(1-\delta)(z_n-x^*)+\delta\sum_{i\geq 1}\sigma_{i,n}(T(t_i)y_n-x^*)\|^q\\
&\leq \alpha_n\|u-x^*\|^q+(1-\alpha_n)(1-\delta)\|z_n-x^*\|^q+(1-\alpha_n)\delta\|y_n-x^*\|^q\\
&\leq \alpha_n\|u-x^*\|^q+(1-\alpha_n)(1-\delta)\|z_n-x^*\|^q\\
&\quad +(1-\alpha_n)\delta\Big[\|z_n-x^*\|^q+\lambda(d_q\lambda^{q-1}-q\alpha)\|Az_n-Ax^*\|^q\Big]\\
&\leq \alpha_n\|u-x^*\|^q+\|z_n-x^*\|^q+(1-\alpha_n)\delta\lambda(d_q\lambda^{q-1}-q\alpha)\|Az_n-Ax^*\|^q.
\end{aligned}$$

This implies that

$$(1-\alpha_n)\delta\lambda(q\alpha-d_q\lambda^{q-1})\|Az_n-Ax^*\|^q \;\leq\; \alpha_n\|u-x^*\|^q$$

Thus, since $\{z_n\}_{n\geq 1}$ is bounded, $\alpha_n\to 0$ as $n\to\infty$, we have that $\lim\limits_{n\to\infty}\|Az_n-Ax^*\|=0$. Besides, by property (10) of Q and Lemma 2.5, we obtain

$$\begin{aligned}
\|y_n-x^*\|^q &= \|Q(z_n-\lambda Az_n)-Q(x^*-\lambda Ax^*)\|^q\\
&\leq \big\langle (z_n-\lambda Az_n)-(x^*-\lambda Ax^*), j_q(y_n-x^*)\big\rangle\\
&= \big\langle z_n-x^*, j_q(y_n-x^*)\big\rangle-\lambda\big\langle Az_n-Ax^*, j_q(y_n-x^*)\big\rangle\\
&\leq \frac{1}{q}\Big(\|y_n-x^*\|^q+d_q\|z_n-x^*\|^q-\|y_n-z_n\|^q\Big)\\
&\quad -\lambda\big\langle Az_n-Ax^*, j_q(y_n-x^*)\big\rangle,
\end{aligned}$$

so that

$$\|y_n-x^*\|^q \leq \frac{d_q}{q-1}\|z_n-x^*\|^q-\frac{1}{q-1}\|y_n-z_n\|^q-\frac{q\lambda}{q-1}\big\langle Az_n-Ax^*, j_q(y_n-x^*)\big\rangle.$$

Thus, we have

$$\begin{aligned}
\|z_n - x^*\|^q &= \|\alpha_n(u-x^*) + (1-\alpha_n)(1-\delta)(z_n - x^*) + \delta \sum_{i\geq 1} \sigma_{i,n}(T(t_i)y_n - x^*)\|^q \\
&\leq \alpha_n\|u-x^*\|^q + (1-\alpha_n)(1-\delta)\|z_n - x^*\|^q + (1-\alpha_n)\delta\|y_n - x^*\|^q \\
&\leq \alpha_n\|u-x^*\|^q + (1-\alpha_n)(1-\delta)\|z_n - x^*\|^q \\
&\quad +(1-\alpha_n)\delta\Big[\frac{d_q}{q-1}\|z_n - x^*\|^q - \frac{1}{q-1}\|y_n - z_n\|^q \\
&\quad -\frac{q\lambda}{q-1}\big\langle Az_n - Ax^*, j_q(y_n - x^*)\big\rangle\Big] \\
&= \alpha_n\|u-x^*\|^q + (1-\alpha_n)\Big[1-\delta\Big(1-\frac{d_q}{q-1}\Big)\Big]\|z_n - x^*\|^q \\
&\quad -\frac{\delta(1-\alpha_n)}{q-1}\|y_n - z_n\|^q - \frac{q\delta\lambda(1-\alpha_n)}{q-1}\big\langle Az_n - Ax^*, j_q(y_n - x^*)\big\rangle.
\end{aligned}$$

But $q \geq 1 + d_q$. Thus,

$$\frac{\delta(1-\alpha_n)}{q-1}\|y_n - z_n\|^q \;\leq\; \alpha_n\|u-x^*\|^q - \frac{q\delta\lambda(1-\alpha_n)}{q-1}\big\langle Az_n - Ax^*, j_q(y_n - x^*)\big\rangle.$$

Using the fact that $\alpha_n \to 0$ and $\|Az_n - Ax^*\| \to 0$ as $n \to \infty$, we obtain that

$$\lim_{n\to\infty} \|y_n - z_n\| = 0 \text{ and thus, } y_n \to u^* \text{ as } n \to \infty.$$

Let

$$Mv = \begin{cases} Av + N_K v, \ v \in K, \\ \emptyset, \ v \notin K. \end{cases}$$

Then, M is maximal accretive. Let $G(M)$ denote the graph of M. Let $(v,w) \in G(M)$. Since $w - Av \in N_K v$ and $y_n \in K$, we have (by definition of $N_K v$) that $\big\langle w - Av, j_q(v - y_n)\big\rangle \geq 0$. Also by the definition of $y_n,\ n \in \mathbb{N}$ and property (9) of Q, we have that

$$\big\langle y_n - (z_n - \lambda Az_n), j_q(v - y_n)\big\rangle \geq 0.$$

Hence,

$$\Big\langle \frac{y_n - z_n}{\lambda} + Az_n, j_q(v - y_n)\Big\rangle \geq 0.$$

Using this, we obtain the following estimate:

$$\begin{aligned}
\big\langle w, j_q(v - y_n)\big\rangle &\geq \big\langle Av, j_q(v - y_n)\big\rangle \\
&\quad \geq \big\langle Av, j_q(v - y_n)\big\rangle - \Big\langle \frac{y_n - z_n}{\lambda} + Az_n), j_q(v - y_n)\Big\rangle \\
&= \Big\langle Av - \frac{y_n - z_n}{\lambda} - Az_n, j_q(v - y_n)\Big\rangle \\
&= \big\langle Av - Ay_n, j_q(v - y_n)\big\rangle + \big\langle Ay_n - Az_n, j_q(v - y_n)\big\rangle \\
&\quad -\Big\langle \frac{y_n - z_n}{\lambda}, j_q(v - y_n)\Big\rangle \\
&\geq \alpha\|Av - Ay_n\| + \big\langle Ay_n - Az_n, j_q(v - y_n)\big\rangle \\
&\quad -\Big\langle \frac{y_n - z_n}{\lambda}, j_q(v - y_n)\Big\rangle \\
&\geq \big\langle Ay_n - Az_n, j_q(v - y_n)\big\rangle - \Big\langle \frac{y_n - z_n}{\lambda}, j_q(v - y_n)\Big\rangle
\end{aligned}$$

so that as $n \to \infty$ (using the fact that A is uniformly continuous and the fact that E has uniformly Gâteaux differentiable norm), we have that

$$\langle w, j_q(v - u^*) \rangle \geq 0,$$

and since M is maximal accretive, we obtain that $u^* \in M^{-1}(0)$. So, by Proposition 2.1 $u^* \in VI(K, A)$ or equivalently $u^* = Q(I - \lambda A)u^*$. Hence, $\{z_n\}_{n \geq 1}$ converges strongly to $u^* \in FV$. □

Remark 3.7. *If E is real L_q (or ℓ_q) space, $2 \leq q < +\infty$, then we have from (2) of Lemma 2.6 that $d_q = q - 1$ so that $1 + d_q = q$. Hence, Theorem 3.6 holds in L_q (or ℓ_q) space, $2 \leq q < +\infty$, In particular, Theorem 3.6 holds in real Hilbert spaces. Thus, we have the following corollaries.*

Corollary 3.8. *Let H be a real Hilbert space. Let $K,\ A,\ \{z_n\}$ and $\{T(t) : t \in \mathbb{R}^+\}$ be as in Theorem 3.6 and $Q = P_K$. Then $\{z_n\}$ converges strongly to an element of FV.*

Corollary 3.9. *Let E be real L_q (or ℓ_q) space, $2 \leq q < +\infty$,. Let $K,\ A,\ \{z_n\}$ and $\{T(t) : t \in \mathbb{R}^+\}$ be as in Theorem 3.6. Then $\{z_n\}$ converges strongly to an element of FV.*

Remark 3.10. *We note that if $1 < q < 2$, then by (1) of Lemma 2.6, $d_q = \frac{1+b_q^{q-1}}{(1+b_q)^{q-1}}$, where b_q is the unique solution of the equation $(q-2)b^{q-1} + (q-1)b^{q-2} - 1 = 0,\ 0 < b < 1$. It therefore follows that $1 + b_q^{q-1} = (q-1)(b_q+1)b_q^{q-2}$. Hence,*

$$\begin{aligned} d_q = \frac{1 + b_q^{q-1}}{(1+b_q)^{q-1}} &= \frac{(q-1)(b_q+1)b_q^{q-2}}{(1+b_q)^{q-1}} \\ &= (q-1)\Big(\frac{b_q}{1+b_q}\Big)^{q-2} = (q-1)\Big(\frac{1+b_q}{b_q}\Big)^{2-q} > q - 1 \end{aligned}$$

(since $1 < q < 2$). Thus, $q < 1 + d_q$. So Theorem 3.6 is not applicable in L_q (or ℓ_q) spaces when $1 < q < 2$.

4 Applications.

We now construct an explicit iterative algorithm for approximation of a common element of fixed point set of family of uniformly asymptotic regular nonexpansive semigroup and the set of solutions of variational inequality problem for an α-inverse strongly accretive operator.

Theorem 4.1. *Let K be a nonempty closed, convex nonepansive retract of a strictly convex q-uniformly smooth real Banach space E with nonexpansive retraction Q and such that $q \geq 1 + d_q$. Let Ψ be a family of uniformly asymptotic regular nonexpansive semigroup from K into itself. Let $A : K \to E$ be an α-inverse strongly accretive mapping. For some real numbers $\delta \in (0,1)$ and $\lambda \in \Big(0, \Big(\frac{q\alpha}{d_q}\Big)^{\frac{1}{q-1}}\Big)$ and for arbitrary $x_1, u \in K$, define a sequence $\{x_n\}_{n=1}^{\infty}$ by*

$$x_{n+1} = \alpha_n u + (1-\delta)(1-\alpha_n)x_n + \delta \sum_{i \geq 1} \sigma_{i,n} T(t_i) Q(x_n - \lambda A x_n),\ n \geq 1. \tag{21}$$

Suppose that $\lim_{n\to\infty} \sum_{i \geq 1} |\sigma_{i,n+1} - \sigma_{i,n}| = 0$, then $\{x_n\}_{n \geq 1}$ converges strongly to some element of FV.

Proof. We first show that $\{x_n\}_{n \geq 1}$ is bounded. Let $x^* \in FV$. We have the following claim.

Claim. $\|x_n - x^*\| \leq \max\{\|u - x^*\|,\ \|x_1 - x^*\|\}\ \forall\, n \geq 1.$

Proof of Claim. We do this by induction. Observe that for $n=1$, the claim clear holds. Suppose that for some $k \in \mathbb{N}\backslash\{1\}$, the claim is true for $n=k$. We show that it is also true for $n=k+1$. Now, we have from equation 21 that

$$\begin{aligned}\|x_{k+1}-x^*\| &\le \alpha_k\|u-x^*\|+(1-\alpha_k)(1-\delta)\|x_k-x^*\| \\ &\quad +\delta\sum_{i\ge1}\sigma_{i,k}\|T(t_i)Q(x_k-\lambda Ax_k)-x^*\| \\ &\le \alpha_k\|u-x^*\|+(1-\alpha_k)\|x_k-x^*\| \\ &\le \max\{\|u-x^*\|,\ \|x_1-x^*\|\}.\end{aligned}$$

So, the claim is true for $n=k+1$. Hence, $\{x_n\}_{n\ge1}$ is bounded. It is now easy to see that $\{T(t_i)Q(x_n-\lambda Ax_n)\}_{n\ge1}$, $\{Ax_n\}_{n\ge1}$ and $\{T(t_i)x_n\}_{n\ge1}$, $i=1,2,...$ are all bounded. Now, define the sequences $\{\gamma_n\}_{n\ge1}$ and $\{u_n\}_{n\ge1}$ by $\gamma_n=(1-\delta)\alpha_n+\delta$ and $u_n=\frac{x_{n+1}-x_n+\gamma_n x_n}{\gamma_n}$. Then

$$u_n=\frac{\alpha_n u+\delta\sum_{i\ge1}\sigma_{i,n}T(t_i)Q(x_n-\lambda Ax_n)}{\gamma_n}.$$

Observe that $\{u_n\}_{n\ge1}$ is bounded and that

$$\begin{aligned}\|u_{n+1}-u_n\|-\|x_{n+1}-x_n\| &\le \left|\frac{\alpha_{n+1}}{\gamma_{n+1}}-\frac{\alpha_n}{\gamma_n}\right|\|u\|+\left|\frac{\delta(1-\alpha_{n+1})}{\gamma_{n+1}}-1\right|\|x_{n+1}-x_n\| \\ &\quad +\frac{\delta M_0}{\gamma_{n+1}\gamma_n}\Big(\sum_{i\ge1}|\sigma_{i,n+1}-\sigma_{i,n}|+|\gamma_{n+1}-\gamma_n|\Big),\end{aligned}$$

for some real constant $M_0>0$. This implies that

$$\limsup_{n\to\infty}(\|u_{n+1}-u_n\|-\|x_{n+1}-x_n\|)\le 0.$$

So, by lemma 2.4, $\lim_{n\to0}\|u_n-x_n\|=0$. But,

$$\|x_{n+1}-x_n\|=\gamma_n\|u_n-x_n\|.$$

Thus,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.$$

Moreover, from the recursion formula (21) we have that

$$x_{n+1}-x_n=\alpha_n(u-x_n)+\delta\sum_{i\ge1}\sigma_{i,n}(T(t_i)Q(x_n-\lambda Ax_n)-x_n).$$

This implies that

$$\delta\Big\|\sum_{i\ge1}\sigma_{i,n}(T(t_i)Q(x_n-\lambda Ax_n)-x_n)\Big\|\le\|x_{n+1}-x_n\|+\alpha_n\|u-x_n\|.$$

Thus,

$$\lim_{n\to\infty}\Big\|\sum_{i\ge1}\sigma_{i,n}(T(t_i)Q(x_n-\lambda Ax_n)-x_n)\Big\|=0.$$

Let $\{\alpha_n\}_{n\ge1}$ in Remark 3.2 also satisfy the following condition:

$$\lim_{n\to\infty}\frac{\|\sum_{i\ge1}\sigma_{i,n}(T(t_i)Q(x_n-\lambda Ax_n)-x_n)\|}{\alpha_n}=0.$$

For each $n \in \mathbb{N}$, let $z_n \in K$ be the unique fixed point satifying (13). Then, by Theorem 3.6, we have that $z_n \to u^* \in FV$ as $n \to \infty$. Using (13) and Lemma 2.2 we obtain that

$$\begin{aligned}
\|z_n - x_n\|^2 &\leq \|(1-\delta)(1-\alpha_n)(z_n - x_n) + \delta\sum_{i\geq 1}\sigma_{i,n}\Big(T(t_i)Q(z_n - \lambda A z_n) \\
&\quad - T(t_i)Q(x_n - \lambda A x_n) + T(t_i)Q(x_n - \lambda A x_n) - x_n\Big)\|^2 \\
&\quad + 2\alpha_n\big\langle u - x_n, j(z_n - x_n)\big\rangle \\
&\leq \Big[(1-\delta)(1-\alpha_n)\|z_n - x_n\| + \delta(1-\alpha_n)\|z_n - x_n\| \\
&\quad + \|\delta\sum_{i\geq 1}\sigma_{i,n}\Big(T(t_i)Q(x_n - \lambda A x_n) - x_n\Big)\|\Big]^2 \\
&\quad + 2\alpha_n\big\langle u - x_n, j(z_n - x_n)\big\rangle \\
&= \Big[(1-\alpha_n)\|z_n - x_n\| + \delta\|\sum_{i\geq 1}\sigma_{i,n}\Big(T(t_i)Q(x_n - \lambda A x_n) - x_n\Big)\|\Big]^2 \\
&\quad + 2\alpha_n\big\langle u - x_n, j(z_n - x_n)\big\rangle.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\big\langle u - x_n, j(x_n - z_n)\big\rangle \leq \frac{\alpha_n}{2}\|z_n - x_n\|^2 \\
&+\delta(1-\alpha_n)\|z_n - x_n\|\Big(\frac{\|\sum_{i\geq 1}\sigma_{i,n}(T(t_i)Q(x_n - \lambda A x_n) - x_n)\|}{\alpha_n}\Big) \\
&+\frac{\delta^2\|\sum_{i\geq 1}\sigma_{i,n}(T(t_i)Q(x_n - \lambda A x_n) - x_n)\|^2}{2\alpha_n}.
\end{aligned}$$

Hence,

$$\limsup_{n\to\infty}\big\langle u - x_n, j(x_n - z_n)\big\rangle \leq 0.$$

Furthermore,

$$\begin{aligned}
\big\langle u - x_n, j(x_n - z_n)\big\rangle &= \big\langle u - u^*, j(x_n - u^*)\big\rangle \\
&\quad + \big\langle u - u^*, j(x_n - z_n) - j(x_n - u^*)\big\rangle \\
&\quad + \big\langle u^* - z_n, j(x_n - z_n)\big\rangle.
\end{aligned}$$

Again, using the fact that E has uniformly Gâteaux differentiable norm we have that

$$\limsup_{n\to\infty}\big\langle u - u^*, j(x_n - u^*)\big\rangle \leq 0.$$

Set

$$\omega_n = \max\{0,\ \big\langle u - u^*, j(x_n - u^*)\big\rangle\}.$$

Then, $0 \leq \omega_n\ \forall\ n \geq 1$. Also, if $\varepsilon > 0$ is given, then there exists $n_\varepsilon \in \mathbb{N}$ such that $\big\langle u - u^*, j(x_n - u^*)\big\rangle < \varepsilon\ \forall\ n \geq n_\varepsilon$. This implies that $0 \leq \omega_n < \varepsilon\ \forall\ n \geq n_\varepsilon$. Thus, $\lim\limits_{n\to\infty}\omega_n = 0$. Moreover, we have from the recursion formula (21) and Lemma 2.2 that

$$\begin{aligned}
\|x_{n+1} - u^*\|^2 &= \|\alpha_n(u - u^*) + (1-\alpha_n)(1-\delta)(x_n - u^*) \\
&\quad + \delta\sum_{i\geq 1}\sigma_{i,n}(T(t_i)Q(x_n - \lambda A x_n) - u^*)\|^2 \\
&\leq \|(1-\alpha_n)(1-\delta)(x_n - u^*) \\
&\quad + \delta\sum_{i\geq 1}\sigma_{i,n}(T(t_i)Q(x_n - \lambda A x_n) - u^*)\|^2 \\
&\quad + 2\alpha_n\big\langle u - u^*, j(x_{n+1} - u^*)\big\rangle \\
&\leq (1-\alpha_n)\|x_n - u^*\|^2 + 2\alpha_n\omega_{n+1}.
\end{aligned}$$

Hence, by Lemma 2.3, we have that $\{x_n\}_{n\geq 1}$ converges strongly to $u^* \in FV$. This completes the proof. □

Corollary 4.2. *Let E be a strictly convex q-uniformly smooth real Banach space such that $q \geq 1 + d_q$. Let $A : E \to E$ be an α-inverse strongly accretive mapping. Let Ψ be a family of uniformly asymptotic regular nonexpansive semigroup from E into itself such that $F(\Psi) \bigcap A^{-1}(0) \neq \emptyset$. For fixed $u \in E$, let $\{x_n\}_{n\geq 1}$ be a sequence generated from arbitrary $x_1 \in E$ by*

$$x_{n+1} = \alpha_n u + (1-\delta)(1-\alpha_n)x_n + \delta \sum_{i\geq 1} \sigma_{i,n} T(t_i)(x_n - \lambda A x_n), \ n \geq 1, \tag{22}$$

where $\{\alpha_n\}_{n\geq 1}$ and $\{\sigma_{i,n}\}_{n\geq 1}$, $i = 1, 2, \ldots$ are as in Theorem 4.1. Then, $\{x_n\}_{n\geq 1}$ converges strongly to an element of $F(\Psi) \bigcap A^{-1}(0)$.

Proof. Here, $A^{-1}(0) = VI(E, A)$, $FV := F(\Psi) \bigcap A^{-1}(0)$ and $Q = I$, where I is the identity operator defined on E. The rest follows as in the proof of Theorem 4.1. □

Corollary 4.3. *Let H be a real Hilbert space. Let $A : H \to H$ be an α-inverse strongly monotone mapping. Let Ψ be a family of uniformly asymptotic regular nonexpansive semigroup from H into itself such that $F(\Psi) \bigcap A^{-1}(0) \neq \emptyset$. For fixed $u \in H$, let $\{x_n\}_{n\geq 1}$ be a sequence generated from arbitrary $x_1 \in H$ by 22. Then, $\{x_n\}_{n\geq 1}$ converges strongly to an element of $F(\Psi) \bigcap A^{-1}(0)$.*

Proof. Here, $A^{-1}(0) = VI(H, A)$, $FV := F(\Psi) \bigcap A^{-1}(0)$ and $Q = P_H = I$, where I is the identity operator defined on H. The rest follows as in Theorem 4.1. □

Corollary 4.4. *Let K be a nonempty closed convex subset of a real Hilbert space. Let $S : K \to K$ be a k-strictly pseudocontractive mapping. Let Ψ be a family of uniformly asymptotic regular nonexpansive semigroup from K into itself such that $F(\Psi) \bigcap F(S) \neq \emptyset$. For fixed $u \in K$, let $\{x_n\}_{n\geq 1}$ be a sequence generated from arbitrary $x_1 \in K$ by*

$$x_{n+1} = \alpha_n u + (1-\delta)(1-\alpha_n)x_n + \delta \sum_{i\geq 1} \sigma_{i,n} T(t_i)\Big((1-\lambda)x_n + \lambda S x_n\Big), \ n \geq 1,$$

where $\{\alpha_n\}_{n\geq 1}$ and $\{\sigma_{i,n}\}_{n\geq 1}$, $i = 1, 2, \ldots$ are as in Theorem 4.1. Then, $\{x_n\}_{n\geq 1}$ converges strongly to an element of $F(\Psi) \bigcap F(S)$.

Proof. Put $A = I - S$, where I is the identity operator defined on K. Then A is α-inverse strongly monotone with $\alpha = \frac{1-k}{2}$. Moreover $F(S) = VI(K, A)$, $Q = P_K$ and $P_K(x_n - \lambda A x_n) = (1-\lambda)x_n + \lambda S x_n$. So, by Theorem 4.1, the result follows. □

5 Numerical simulations/sensitivity analysis

Remark 5.1. *Prototypes for our iteration parameters $\{\alpha_n\}_{n\geq 1}$ and $\{\sigma_{i,n}\}_{n\geq 1}$, $i = 1, 2, \ldots$ are*

$$\alpha_n = \frac{1}{n+1}, \ \sigma_{i,n} = \frac{n}{2^i(n+1)}, \ i, \ n \in \mathbb{N}.$$

Remark 5.2. *The addition of bounded error terms in any of the recursion formulas leads to no further generalization.*

Remark 5.3. *If $f : K \to K$ is a contraction map and we replace u by $f(x_n)$ in the recursion formulas of our theorems, we obtain what some authors now call* viscosity *iteration process. We observe that all our theorems in this paper carry over trivially to the so-called viscosity process. One simply replaces u by $f(x_n)$, repeats the argument of this paper, using the fact that f is a contraction map.*

6 Discussions and Conclusion

The theorems obtained in this paper extend, improve and generalize many recently announced results.

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