International Journal of Mathematical Analysis and Modelling

(Formerly Journal of the Nigerian Society for Mathematical Biology)

Volume 7 Issue 1, 2024

ISSN (Print): 2682 - 5694
ISSN (Online): 2682 – 5708
www.tnsmb.org

Type II Half Logistic Inverse Exponential distribution: properties and application to tax data

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Abstract

We defined, studied and inventive distribution called Type II half logistic Inverse Exponential ($TIIHL-IE$) distribution. Some well-known mathematical properties: moments, Bonferroni and Lorenz curves, mean deviation, quantile function, Reliability function, and Renyi entropy of $TIIHL-IE$ distribution is studied. The expressions of order statistics are derived. Parameters of the derived distribution are obtained using maximum likelihood method. The applicability of proposed distribution is exemplified by using a tax revenue dataset.

Keywords and phrases Lorenz curves, quantile function, Reliability function, Mean deviation.

1 Introduction

In the recent decade, there has been an overwhelming demand of probability distribution with a more adaptability properties in several fields of study. This has enabled the development of many generalized distributions. Many of these distributions have been developed by the addition of one or more shape parameters to baseline distribution that exist in literature to make them more flexible and adaptable in modeling various kinds of data exhibiting different shapes of the hazard function. These distributions have to found applications area such as in reliability studies, seismography, insurance, medicine among several others because of their flexibility in modeling failure rate. For example, the Kumaraswamy odd log-logistic family by [1], Weibull-G family of distributions by [3], Kumaraswamy generalized family was studied by [4], exponentiated generalized class of distribution was studied by [5], exponentiated half-logistic was introduced by [6], the type I half-logistic by [7], [8] proposed and developed the Zografos-Balakrishnan odd log-logistic family of distributions,], [9] developed the Zografos Balakrishnan odd log-logistic family of distributions, [10] developed the generalized beta family of distributions. the Kumaraswamy Weibull proposed and developed by [11, a new generalized odd log-logistic family by [12], the kummer beta was developed and studied by [13] among many

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others. This study further extends the inverse exponential distribution to further enhance its scope of applications.

A random variable X follows an Inverse exponential distribution, if its cumulative distribution function (CDF) can be represented as

$$F(x;\omega) = \exp(-\omega x^{-1}), \tag{1}$$

and the related probability density function (PDF) to (1) is

$$F(x;\omega) = \omega x^{-2}\exp(-\omega x^{-1}), x; \omega, \alpha > 0. \tag{2}$$

where w is the scale parameter.

1.1 Motivation of study

The chief motivation for this study is to exploy the flexibility offers by the Type II half logistic-G family of distribution to any baseline distribution, which makes the baseline distribution to be more flexible and adaptable to any kind of data exhibiting any form of the shapes of the failure rate. On this ground we are proposing the Type II half logistic Inverse Exponential ($TIIHL - IE$) distribution.

2 The $TIIHL - IE$ Distribution

[11] proposed and developed Type II half logistic-G (TIIHL-G) family of distributions with the CDF given by

$$G(x;\alpha,\varphi) = 1 - \int_0^{-logF(x;\varphi)} \frac{2\alpha e^{-\alpha t}}{(1+e^{-\alpha t})^2}dt \tag{3}$$

$$= \frac{2[F(x;\varphi)]^\alpha}{1+[F(x;\varphi)]^\alpha}, \qquad x > 0, \alpha > 0,$$

where $F(x;\varphi)$ the CDF of baseline model with parameter is vector φ and $G(x;\alpha,\varphi)$ is CDF obtained by using the T-X generator developed by [2]. The PDF of the $TIIHL - G$ family is given by

$$g(x;\alpha,\varphi) = \frac{2\alpha f(x;\varphi)[F(x;\varphi)]^{\alpha-1}}{[1+[F(x;\varphi)]^\alpha]^2}, \qquad x > 0, \theta > 0, \tag{4}$$

A random variable (r.v) X follows Type II half logistic Frechet ($TIIHL - IE$) distribution if its CDF is obtained by inserting (1) in (3) as follows:

$$G(x;\alpha,\omega) = \frac{2\left[e^{-\omega x^{-1}}\right]^\alpha}{1+\left[e^{-\omega x^{-1}}\right]^\alpha}, x > 0; \omega, \alpha > 0 \tag{5}$$

The PDF corresponding to (5) is given by

$$g(x;\alpha,\omega) = 2\alpha\omega x^{-2}e^{-\omega\alpha x^{-1}}\left(1+e^{-\omega\alpha x^{-1}}\right)^{-2}, x > 0; \omega, \alpha > 0 \tag{6}$$

An expression for the reliability function and the failure rate function is given respectively as

$$R(x;\alpha,\omega) = 1 - \frac{2\left[e^{-\omega x^{-1}}\right]^\alpha}{1+[e^{-\alpha x^{-\alpha}}]^\alpha}. \tag{7}$$

and

$$h(x;\alpha,\omega)=\frac{2\alpha\omega\alpha x^{-2}e^{-\omega x^{-2}}\left(1+e^{-\omega\alpha x^{-1}}\right)^{-1}}{1-e^{-\omega\alpha x^{-1}}}\ . \tag{8}$$

Figures 1 and 2 demonstrate the flexibility of the $TIIHL-IE$ distribution. The plots for the PDF of $TIIHL-IE$ distribution is given in figure 1, and plot of the hazard rate function (hrf) is given in Figure 2 for various values of the parameters

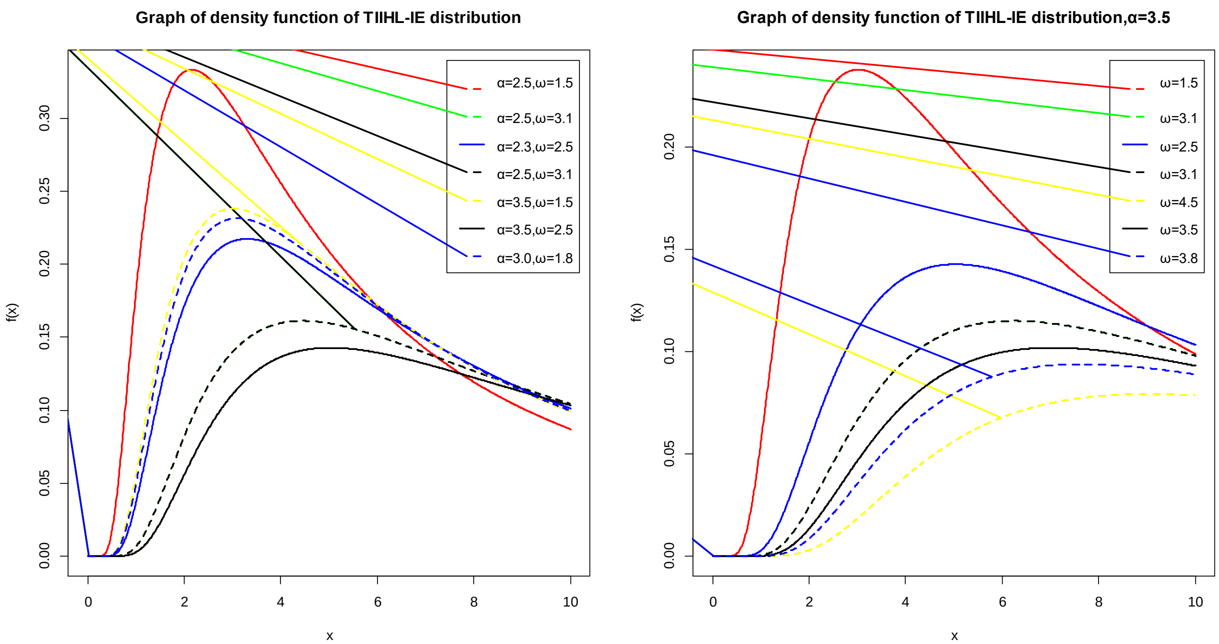

Figure 1. Plots of the PDF of $TIIHL-IE$ distribution for selected values of the parameters

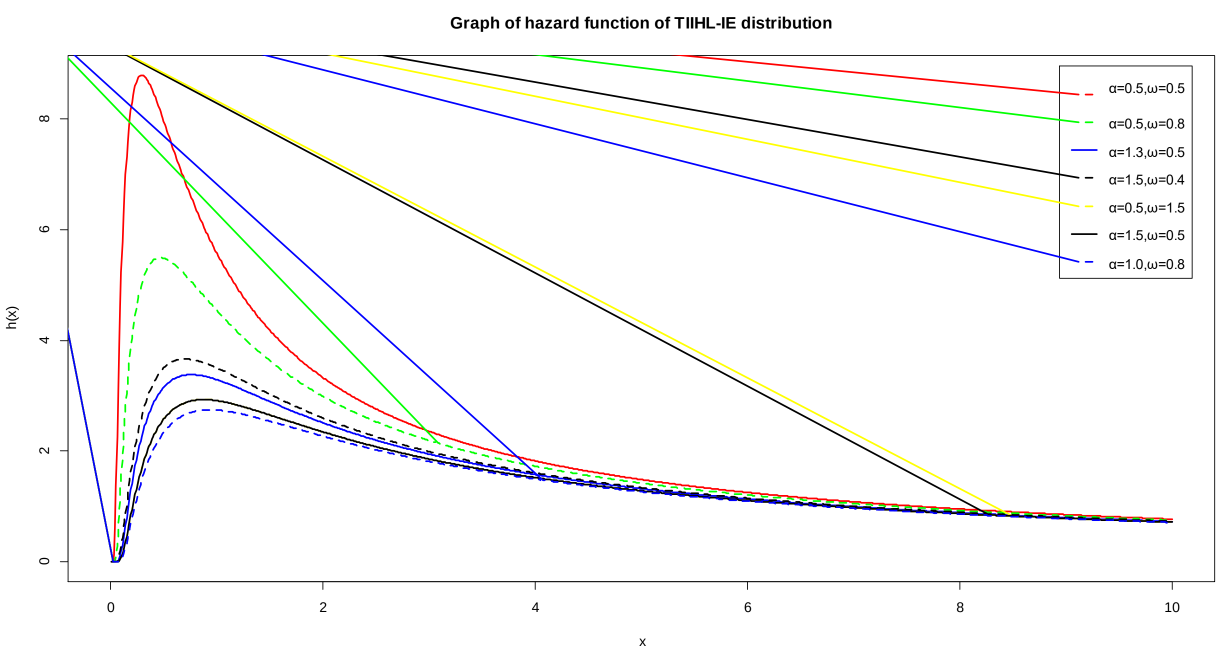

Figure 2. Plots of the PDF of $TIIHL-IE$ distribution for selected values of the parameters

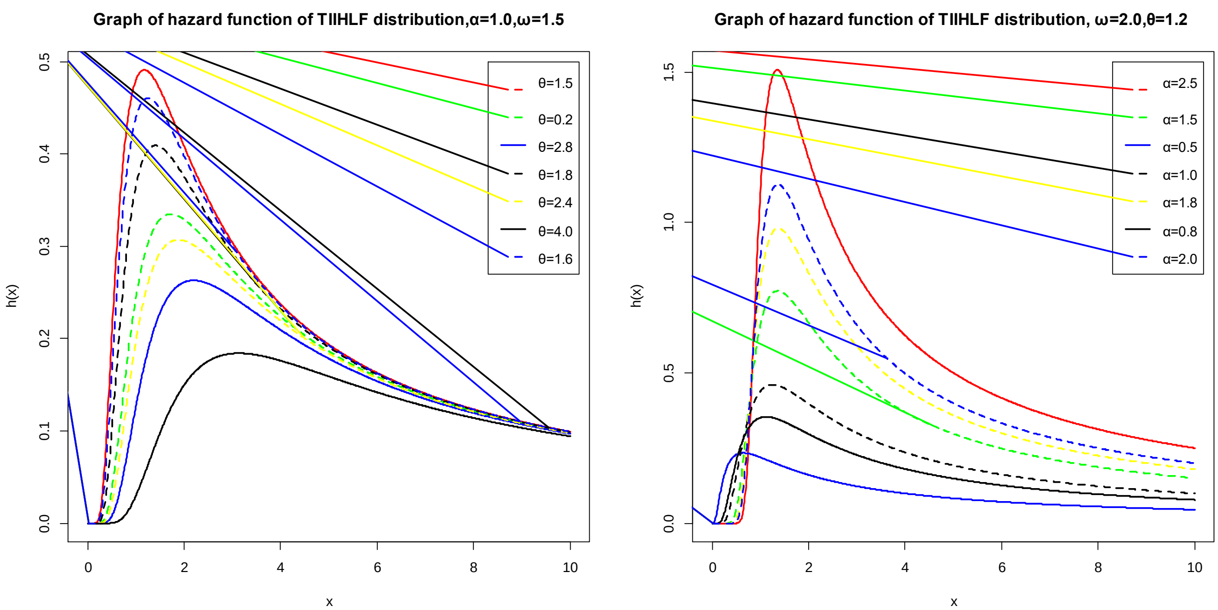

Figure 3. Plots of the hf of TIIHL-IE distribution for selected values of the parameters

subsections

3 Some Mathematical Properties

Here, we derived some properties of $TIIHL-IE$ distribution.

3.1 Quantile function.

The quantile function of X is denoted by $Q(u)$, defined as

$$Q(u) = F^{-1}\left(\frac{u}{2-u}\right)^{1/\theta}, u \in (0,1), \tag{9}$$

The inverse function of $F(.)$ is $F^{-1}(.)$, given in (1)
Also,

$$x_u = \left[-\frac{1}{\omega\alpha}\left(ln\left\{\frac{u}{2-u}\right\}\right)\right]^{-1} . \tag{10}$$

It should be noted that simulated values from the quantile function of $TIIHL-IE$ distribution given in (10) can be utilized to demonstrates the effectiveness of the method of estimation approach adopted. The random variable q follows a uniform distribution on the interval $0,1$; i.e $x_u = Q(u)$ follows $TIIHL-IE$ distribution in particular, the lower, middle and upper quartiles are obtained by taking $u = 0.25, 0.5$ and 0.75 respectively in (10).

3.2 Alternate representation.

Here, we present the expansion of the PDF and CDF for $TIIHL-IE$ distribution for further mathematical simplification. The binomial theorem, for $z>0$ and $|m|<1$, can be expressed as

$$(1+y)^{-z}=\sum_{i=0}^{\infty}(-1)^{i}\binom{z+i-1}{i}y^{i}, \tag{11}$$

Then, by applying (11) in (6), the PDF of $TIIHL-IE$ distribution becomes

$$g(x)=2\alpha\omega\sum_{i=0}^{\infty}(-1)^{i}\binom{i+1}{i}x^{-2}e^{-\alpha\omega(i+1)x^{-1}} \tag{12}$$

Proposition 1. Let "s" be a positive integer. The expansion of the CDF can be written in the form:

$$[F(x)]^{s}=2^{s}\sum_{j=0}^{\infty}(-1)^{j}\binom{s+j-1}{j}e^{-\omega\alpha(j+s)x^{-2}}, \tag{13}$$

Proof. The CDF $[F(x)]^{s}$ is obtained, for 's' an integer by using (12)

$$[F(x)]^{s}=\left[\frac{2\left[e^{-\omega x^{-1}}\right]^{\alpha}}{1+\left[e^{-ax^{-1}}\right]^{\alpha}}\right]^{s},$$

$$[F(x)]^{s}=2^{s}\sum_{j=0}^{\infty}(-1)^{j}\binom{s+j-1}{j}e^{-\omega\alpha(j+s)x^{-2}} \tag{14}$$

3.3 Moments of $TIIHL-IE$ distribution

The moment plays an important role in distribution study and real data application(s). Now, we obtain the k^{th} moment of the $TIIHL-IE$ distribution under certain regularity condition; the k^{th} moment of $TIIHL-IE$ distribution is obtained as

$$\mu_k'=\int_{-\infty}^{\infty}x^{k}f(x)dx=2\alpha\omega\theta\sum_{i=0}^{\infty}(-1)^{i}\binom{i+1}{i}Q(x) \tag{15}$$

where

$$Q(x)=\int_{-\infty}^{\infty}x^{k-2}\,e^{-\omega\alpha(i+1)x^{-\alpha}}dx \tag{16}$$

taking $p=\omega\alpha(i+1)x^{-1}$,$dx=-\frac{p^{-2}}{[\omega\alpha(i+1]^{-1}}dp$ and substitute in (16), we have

$$Q(x)=\frac{1}{[\omega\alpha(i+1]^{1-k}}\int_{0}^{\infty}p^{-k}e^{-p}dp=\frac{1}{[\omega\alpha(i+1]^{1-k}}\Gamma(1-k)$$

Finally, we have an expression for the kth moment of $TIIHL-IE$ distribution given as

$$\mu_k'=2\omega\alpha\sum_{i=0}^{\infty}(-1)^{i}\binom{i+1}{i}\frac{1}{[\omega\alpha(i+1]^{1-k}}\Gamma(1-k) \tag{17}$$

3.4 Incomplete moments of $TIIHL-IE$ distribution

The incomplete moment of $TIIHL-IE$ distribution can be obtained using (13) as

$$\varphi_k(t) = \int_{-\infty}^{t} x^k f(x)dx = 2\alpha\omega \sum_{i=0}^{\infty} (-1)^i \binom{i+1}{i} M(x) \tag{18}$$

where

$$M(x) = \int_{-\infty}^{t} x^{k-2}\, e^{-\omega\alpha(i+1)x^{-1}} dx \tag{19}$$

taking $= \omega\alpha(i+1)x^{-1}$,$dx = -\frac{p^{-2}}{[\omega\alpha(i+1]^{-1}} dp$ and substitute in (19), we have

$$M(x) = \frac{1}{[\omega\alpha(i+1]^{1-k}} \int_{0}^{t} p^{-k/\alpha} e^{-p} dp = \frac{1}{[\omega\alpha(i+1]^{1-k}} \Gamma(1-k, \omega\alpha(i+1)t^{-1})$$

Where

Finally, we have

$$\varphi_k(t) = 2\omega\alpha \sum_{i=0}^{\infty} (-1)^i \binom{i+1}{i} \frac{1}{[\omega\alpha(i+1]^{1-k}} \Gamma(1-k, \omega\alpha(i+1)t^{-1}) \tag{20}$$

For $k = 1, 2, 3, \ldots.$

3.5 Moment generating function of $TIIHL - IE$ distribution

The moment generating function of a random variable $X\ TIIHL - IE$ distribution is obtained as

$$M_X(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mu_k' = 2\omega\alpha \sum_{i,k=0}^{\infty} (-1)^i \binom{i+1}{i} \frac{t^k}{k!} \frac{1}{[\omega\alpha(i+1]^{1-k}} \Gamma(1-k) \tag{21}$$

The mean deviation: the expression to obtain the mean deviation about the mean and median are respectively given by
$M_1(x) = 2\mu F(\mu) - 2\phi(\mu)$ and $M_2(x) = 2\mu F(\mu) - 2\phi(M_d)$, where M_d is the median of X and $\phi(q) = \int_{-\infty}^{q} xf(x)dx$ is the incomplete moment given in (21).
Now

$$T(\mu) = \int_{0}^{\mu} xf(x)dx = \sum_{i=0}^{\infty} (-1)^i \binom{i+1}{i} \frac{1}{[\omega\alpha(i+1]^{1-k/\alpha}} \Gamma(1-k, \omega\alpha(i+1)\mu^{-1})$$

$$T(M_d) = \int_{0}^{M_d} xf(x)dx = \sum_{i=0}^{\infty} (-1)^i \binom{i+1}{i} \frac{1}{[\omega\alpha(i+1]^{1-k}} \Gamma\left(1-k, \omega\alpha(i+1){M_d}^{-1}\right)$$

3.6 Bonferroni and Lorenz curves

Bonferroni and Lorenz curves are mostly applied in different fields of life such as demography studies, medical technology, life insurance, life testing, and reliability analysis among others. The Lorenz and Bonferroni Curves are computed using the expression from the first incomplete moment and can respectively be obtained as follows:

$$L_F(x) = \frac{1}{E(X)}\int_0^x tf(t)dt = \frac{\sum_{i=0}^{\infty}(-1)^i\binom{i+1}{i}\frac{1}{[\omega\alpha(i+1]^{1-k}}\Gamma(1-k,\omega\alpha(i+1)t^{-1})}{\sum_{i=0}^{\infty}(-1)^i\binom{i+1}{i}}$$

and

$$B_F(x) = \frac{L_F(x)}{F(x)} = \frac{\sum_{i=0}^{\infty}(-1)^i\binom{i+1}{i}\frac{1}{[\omega\alpha(i+1]^{1-k/\alpha}}\Gamma(1-k,\omega\alpha(i+1)t^{-1})\,/\sum_{i=0}^{\infty}(-1)^i\binom{i+1}{i}}{2e^{-\omega\alpha x^{-1}}/(1+e^{-\omega\alpha x^{-1}})}$$

3.7 Probability weighted moment

The probability weighted moments (PWMs) of a distribution are further used to study some characteristics of probability distribution. Denoted by $\eta_{k,s}$, can be defined as

$$\eta_{k,s} = E[X^k F(X)^s] = \int_{-\infty}^{\infty} x^k F(x)^s f(x)dx \tag{22}$$

Considering and expression in (12) and (14) and putting it in (23), we have

$$\eta_{k,s} = 2^{s+1}\alpha\omega\theta\sum_{i,j=0}^{\infty}(-1)^{i+j}\binom{i+1}{i}\binom{j+s-1}{j}\int_{-\infty}^{\infty} x^{k-\alpha-1}e^{-\omega\alpha x^{-1}(i+j+s+1)}dx \tag{23}$$

Letting $m = \omega\alpha x^{-1}(i+j+s+1)$ and putting it (23), we have

$$\eta_{k,s} = 2^{s+1}\omega\alpha\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}\binom{i+1}{i}\binom{j+s-1}{j}}{[\omega x^{-\alpha}(i+j+s+1)]^{1-k}}\int_{-\infty}^{\infty} m^{-k}e^{-m)}dm \tag{24}$$

Finally, we have,

$$\eta_{k,s} = 2^{s+1}\omega\alpha\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}\binom{i+1}{i}\binom{j+s-1}{j}}{[\omega\alpha x^{-\alpha}(i+j+s+1)]^{1-k}}\Gamma(1-k) \tag{25}$$

3.8 Renyi entropy

Renyi entropy was proposed by [14]. It can be obtained by

$$I_\lambda(x) = \frac{1}{1-\lambda}logM \tag{26}$$

Where

$$M = \int_{-\infty}^{\infty} f^\lambda(x)dx\,, \qquad \lambda > 0, \lambda \neq 1 \tag{27}$$

Putting (6) in (27), we have

$$M = \int_{-\infty}^{\infty}\left[2\alpha\omega x^{-(\alpha+1)}e^{-\omega\alpha x^{-1}}\left(1+e^{-\omega\alpha x^{-1}}\right)^{-2}\right]^\lambda dx \tag{28}$$

$$= (2\alpha\omega)^\lambda\int_{-\infty}^{\infty} x^{-2\lambda}e^{-\omega\alpha\lambda x^{-\alpha}}\left(1+e^{-\omega\alpha x^{-1}}\right)^{-2\lambda}dx$$

Using binomial expansion given in (12) for the expression above, we have

$$M = (2\alpha\omega)^{\lambda} \sum_{i=0}^{\infty} (-1)^i \binom{2\lambda + 1 - 1}{i} \int_{-\infty}^{\infty} x^{-2\lambda} e^{-\omega\alpha(\lambda+i)x^{-1}} dx \tag{29}$$

Taking $z = \omega\alpha(\lambda + i)x^{-2}$, $x = \frac{z^{-1}}{[\omega\theta(\lambda+i)]^{z^{-1}}}$, $dx = -{}^{1}/_{\alpha} \frac{z^{-2}}{[\omega\alpha(\lambda+i)]^{-1}} dz$ and putting it in (29), we obtain

$$M = \sum_{i=0}^{\infty} (-1)^i \binom{2\lambda + 1 - 1}{i} \frac{2^{\lambda}\omega^{\lambda}\alpha^{\lambda}\alpha^{\lambda-1}}{[\omega\theta(\lambda + i)]^{-\frac{\lambda(\alpha+1)-1}{\alpha}}} \int_{0}^{\infty} z^{\frac{(\lambda-1)(\alpha+1)}{\alpha}} e^{-z} dz \tag{30}$$

$$= \sum_{i=0}^{\infty} (-1)^i \binom{2\lambda + 1 - 1}{i} \frac{2^{\lambda}\omega^{\lambda}\alpha^{\lambda}}{[\omega\theta(\lambda + i)]^{-2\lambda-1}} \Gamma[(\lambda - 1) + 1]$$

Finally, the entropy of TIIHL distribution is given by

$$I_{\lambda}(x) = \frac{1}{1-\lambda} log\left(\sum_{i=0}^{\infty} (-1)^i \binom{2\lambda + 1 - 1}{i} \frac{2^{\lambda}\omega^{\lambda}\alpha^{\lambda}}{[\omega\alpha(\lambda + i)]^{2\lambda-1}} \Gamma[(\lambda - 1) + 1]\right) \tag{31}$$

3.9 Stress-strength reliability

Here we obtain the stress-strength parameter of $TIIHL - IE$ distribution. Suppose X_1 represent the strength of a structure with stress X_2, and if X_1 follows $TIIHL - IE$ (ω_1, α_1) and X_2 follows $TIIHL - IE$ (ω_2, α_2), given that $X_1$and X_2 are statistically independent random variables, then the Stress-strength Reliability (R) of TIIHLIE is obtained as follows:

$$R = P(X_2 < X_1) = \int_{0}^{\infty} f_1(x; \omega_1, \alpha_1) F_2(x; \omega_2, \alpha_2) dx \tag{32}$$

Then we can express (32) as

$$R = 4\omega_1\alpha_1 \sum_{j=0}^{\infty} (-1)^{i+j} \binom{i+1}{i} \int_{0}^{\infty} x^{-1} e^{-[\omega_1\alpha_1(j+1)+\omega_2\alpha_2(i+1)]x^{-1}} dx \tag{33}$$

Taking $v = [\omega_1\alpha_1(j + 1) + \omega_2\alpha_2(i + 1)]x^{-1}$, finally we have

$$R = \frac{4\omega_1\alpha_1}{[\omega_1\alpha_1(j + 1) + \omega_2\alpha_2(i + 1)]} \sum_{i,j=0}^{\infty} (-1)^{i+j} \binom{i+1}{i}$$

3.10 Order statistics

Order statistics is widely applied in statistical theory. Suppose $X_1, X_2, \ldots, X_n$ be a random sample having CDF $F(x)$. Let $x_{1:1}, x_{2:n}, \ldots, x_{n:n}$ be the ordered sample of size n, then the density of r^{th} order statistics is given as

$$f_{r:n}(x) = P \sum_{l=0}^{n-r} (-1)^l \binom{n-r}{l} f(x) F(x)^{l+r-1} \tag{34}$$

where $P = \frac{n!}{(n-r)!r!}$. The PDF of the r^{th} order statistics of $TIIHL - IE$ distribution is obtained by putting (13) and (14) in (34), changing s with $l + r - 1$ followed by simple algebraic manipulation, we have

$$f_{r:n}(x) = P\alpha\omega \sum_{l=0}^{n-r} \sum_{i,j=0}^{\infty} H_l \, x^{-2} e^{-\omega\alpha x^{-1}(i+j+l+r)} \tag{35}$$

where

$$\boldsymbol{H_l} = 2^{l+r}(-1)^{i+j+l} \binom{n-r}{l} \binom{i+1}{i} \binom{l+r+j-2}{j}$$

Consequently, the s^{th} moment of the r^{th} order statistics for $TIIHL - IE$ distribution is given by

$$E(X_{r:n}^{s}) = \int_{-\infty}^{\infty} x^s f_{r:n}(x) dx. \tag{36}$$

By substituting (35) in (36), we have

$$E(X_{r:n}^{s}) = \alpha\omega P \sum_{l=0}^{n-r} \sum_{i,j=0}^{\infty} \boldsymbol{H_l} \int_{-\infty}^{\infty} x^{s-2} e^{-\omega\alpha x^{-1}(i+j+l+r)} dx. \tag{37}$$

Letting $= \omega\alpha x^{-1}(i+j+l+r)$, $x = \frac{y^{-1}}{[\omega\alpha(i+j+l+r)]^{-1}}$, $dx = -\frac{y^{-2}}{\alpha[\omega\theta(i+j+l+r)]^{-1}} dy$ and putting it (39), we have

$$E(X_{r:n}^{s}) = P \sum_{l=0}^{n-r} \sum_{i,j=0}^{\infty} \frac{\omega\alpha \boldsymbol{H_l}}{[\omega\alpha(i+j+l+r)]^{1-s/\alpha}} \int_{-\infty}^{\infty} y^{-s/r} e^{-y} dx \tag{38}$$

Then,

$$E(X_{r:n}^{s}) = P \sum_{l=0}^{n-r} \sum_{i,j=0}^{\infty} \frac{\omega\alpha H_l}{[\omega\alpha(i+j+l+r)]^{1-s/\alpha}} \Gamma(2-s) \tag{39}$$

4 Maximum likelihood estimation

Given a random sample of $x_1, x_2, \dots, x_n$ from the TIIHLF (α, ω) distribution, the likelihood function for $\xi = (\alpha, \omega)$ is

$$L(\underline{x}, \xi) = \prod_{i=1}^{n} 2\alpha\omega x^{-2} e^{-\omega\alpha x^{-1}} \left(1 + e^{-\omega\alpha x^{-1}}\right)^{-2} \tag{40}$$

And the log-likelihood function $logL(\underline{x}, \xi) = l$ is given as

$$l = nlog(2\alpha\omega) + (\alpha - 1) \sum_{i=1}^{n} log\,(x_i) - \omega\alpha \sum_{i=1}^{n} x_i^{-1} - 2 \sum_{i=1}^{n} log\left(1 + e^{-\omega\alpha x_i^{-1}}\right) \tag{41}$$

The element of the score vector $U(\xi) = (U_\alpha, U_\omega)$ are given below

$$U_\omega = \frac{n}{\theta} - \omega \sum_{i=1}^{n} x_i^{-\alpha} + 2 \sum_{i=1}^{n} \left[\frac{\omega x_i^{-\alpha} e^{-\omega\theta x_i^{-\alpha}}}{1 + e^{-\omega\theta x_i^{-\alpha}}} \right] \tag{42}$$

$$U_\alpha = \frac{n}{\omega} - \alpha \sum_{i=1}^{n} x_i^{-1} + 2 \sum_{i=1}^{n} \left[\frac{\alpha x_i^{-1} e^{-\omega\alpha x_i^{-1}}}{1 + e^{-\omega\alpha x_i^{-1}}} \right] \tag{43}$$

The maximum likelihood estimator $\hat{\xi} = (\hat{\alpha}, \hat{\omega})^T$ can be obtained by solving simultaneously the likelihood equations

$$\frac{\partial l(\xi)}{\partial \alpha}\Big|_{\xi=\hat{\xi}} = 0, \quad \frac{\partial l(\xi)}{\partial \omega}\Big|_{\xi=\hat{\xi}} = 0,$$

There is no closed-form expression for the maximum likelihood estimator and its computation can only be handled numerically using a nonlinear optimization algorithm. We can employ the use of iterative techniques such as a Newton–Raphson-type algorithm to obtain the numerical value of $\hat{\xi}$. The observed information matrix used for computing confidence intervals for the model parameters can be obtained from standard maximization routines numerically such as the use of the R functions optim or nlm, the SAS procedure NL Mixed, the Ox function Max BFGS, and many others, to compute the observed information matrix numerically.

5 Application

In this section, two real data sets are used to demonstrate the flexibility and applicability of the Type II half logistic Frechet model over some of some other competitive model. We consider some important measures of goodness-of-fit: Akaike information criterion (AIC), Bayesian information criterion (BIC), the corrected Akaike information criterion (CAIC) and p-value (PV) are used. For the data set, we shall compare the fits of the TIIHL-IE model with other models: the Lomax -Inverse Exponential (L-IE), Inverse Exponential (IE), and Exponential model. The smaller these statistics are, the better the fit except for the p value. However, the better model corresponds to the smaller values of AIC, BIC, CAIC, and larger value of P value (PV) criteria.

The data set is a taxes revenue's data, the values of tax revenue data are 3.7, 3.11, 4.42, 3.28, 3.75, 2.96,3.39, 3.31, 3.15, 2.81, 1.41, 2.76, 3.19, 1.59, 2.17, 3.51, 1.84, 1.61, 1.57, 1.89, 2.74, 3.27, 2.41, 3.09, 2.43, 2.53,2.81, 3.31, 2.35, 2.77, 2.68, 4.91, 1.57, 2.00, 1.17, 2.17, 0.39, 2.79, 1.08, 2.88, 2.73, 2.87, 3.19, 1.87, 2.95, 2.67,4.20, 2.85, 2.55, 2.17, 2.97, 3.68, 0.81, 1.22, 5.08, 1.69, 3.68, 4.70, 2.03, 2.82, 2.50, 1.47, 3.22, 3.15, 2.97, 2.93,3.33, 2.56, 2.59, 2.83, 1.36, 1.84, 5.56, 1.12, 2.48, 1.25, 2.48, 2.03, 1.61, 2.05, 3.60, 3.11, 1.69, 4.90, 3.39, 3.22,2.55, 3.56, 2.38, 1.92, 0.98, 1.59, 1.73, 1.71, 1.18, 4.38, 0.85, 1.80, 2.12, 3.65.

Table 1 present the Exploratory data analysis of the Tax revenue data. Table 2 presents the maximin likelihood estimate (MLEs) and the measures of goodness of fits for the model considered. Figure 4 is the represent the box plot for Tax revenue data. Table 1.0 indicates that the Tax revenue data is positively skewed, with excess kurtosis of 0.18 and under-dispersed.

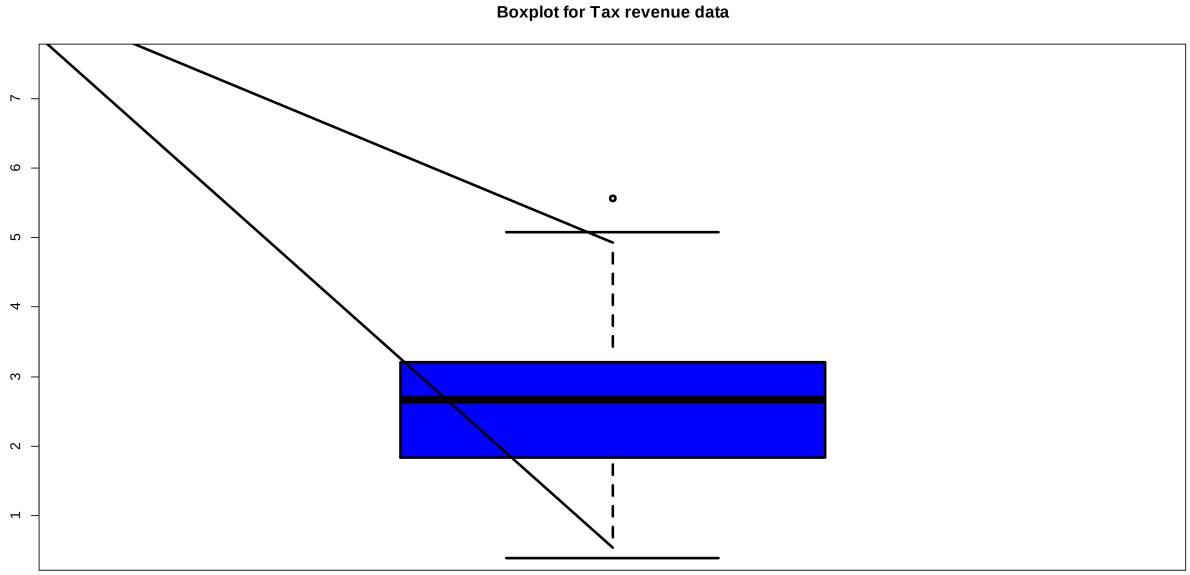

Figure 4: Box plot for Tax revenue data

Table 1.0 Exploratory data Analysis for Tax revenue data

n	Min.	q_1	$median$	q_3	$mean$	$Max.$	$skewness$	$kurtosis$	$variance$
100	0.39	1.84	2.68	3.20	2.61	5.56	0.39	3.18	1.02

Table 2.0: MLEs, standard error (parenthesis), and Measures of goodness of fits

$Model$	$\hat{\alpha}$	$\hat{w}$	$-l$	AIC	$CAIC$	HQIC	BIC
$TIIHLIE$	1.29 (0.27)	1.0901 (0.82)	92.85	189.81	189.82	191.80	194.90
IE	2.1353 (1.98)	$-$ $(-)$	198.98	399.96	299.99	401.01	402.56
E	0.38 (0.04)		195.99	393.98	394.02	395.03	396.58
$L-IE$	2.77 (0.23)	1.67 (0.07)	150.42	304.83	304.96	306.94	310.04

6 Conclusion

We have developed the $TIIHL-IE$ distribution and studied its properties such as: descriptive measures based on the quantiles, moments, stress-strength reliability and incomplete moments and order statistics. Goodness of fit shows that $TIIHL-IE$ distribution provides a better fit. Applications of the TIIHL-IE model to taxes revenue's data demonstrate its significance and flexibility. We have shown that the TIIHL-IE distribution real life data than all other competitive models considered in this study.

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