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Algorithm for semigroup bases I

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Abstract

The study of generating sets and bases in semigroup theory started with the experience in the topics in vector spaces, and so with comparison, theorems on independent generating sets of Semigroup that vary from those about independent generating sets in vector spaces due to some differences in their structures are given. The paper shows that there may exist a maximal independent set of elements in a semigroup which does not form a generating set. It reveals that there may exists independent set in a semigroup which cannot be expanded to a generating set, and that adding an element independent from an independent set may make the independent set dependent in a semigroup. Also, given is an important condition fulfilled by a Basis in a semigroup. The paper gives an interesting property that serves as a relevant test tool for independent generating sets in the developed algorithms for determining the basis of any given semigroup. The resulting algorithm works for determining the basis any semigroup including classes of semigroups.

Keywords and phrases: Basis, Independent, generating set, Minimal, maximal, Vector spaces

1 Introduction

The characterization of an algebraic structure has as one of its popular tasks, the problem of solving for an independent generator of the algebraic structure. Such independent generators are called the bases. So many papers have been written on independent generating sets or bases of semigroups. Some recent ones are [1], [2], [3] and [4]. The paper [2], gives universal solution algorithm for the bases problem of semigroups. The paper [1] discusses certain semigroups in which the number of independent elements is larger than the basis, hence being just independent alone does not guarantee it being the basis of a semigroup. A later paper by [4] improved upon this algorithm which this paper further advances with more theorems that serve as easier tools for implementing the algorithm.,

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It is a common practice to study individual classes of semigroup as part of the numerous efforts to characterize semigroup in general. The approach, II. (in figure 1) which summarizes the solution approach used in this paper for determining the independent generator of semigroups is different from what is often found in most literatures that utilize the Greens equivalences. The green points $t_i, i \in N,$ in figure 1 which represent classes of semigroups are likened to trees of the forest (forming the whole semigroup) while the central spots in each of figures I and II respectively represent the researcher trying to study desired structure of the 'forest of semigroups. The researcher in I analyses from within the 'forest' while that in II. analyses from outside the 'forest' with a clearer view of the entire 'forest structure' but not afforded a detailed understanding of the individual properties of the trees as I. would be more privileged to.

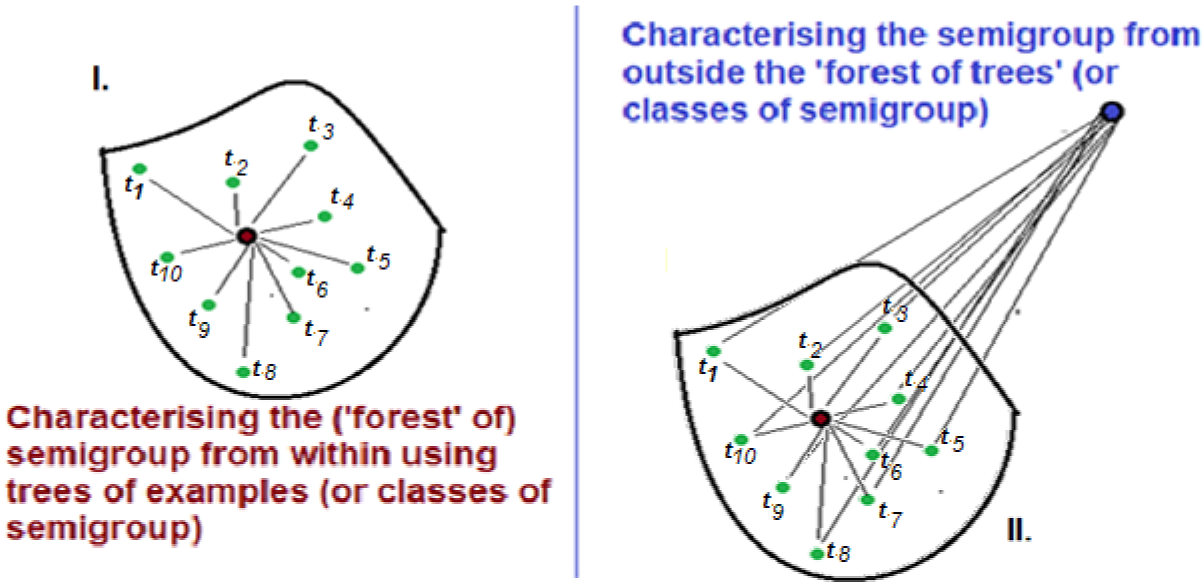

figure 1

The dots $t_i, i \in N$ represent Classes of Semigroups while the central spots represent the researcher I and II respectively.

This work will for now centre on constructing the algorithm for determining the bases of any semigroup. In a later paper, we will present the utilization of the ideas raised here in determining the Basis of classes of semigroup.

Definition 1.1. Let the set $P(W)$ be the power-set of the non-empty set W. A set $A \in P(W)$ with a property pr is maximal (minimal) if $\nexists U \in P(W)$ with the property pr such that $A \subset U$ $(U \subset A)$ and $A \neq U$.

Definition 1.2. The set $H \subset V$ is a generating set if $L(H) =: < H > = V$.

Definition 1.3. Let $\emptyset \neq H \subset V$ then $x \in V$ is independent from H if $x \notin L(H)$.

Definition 1.4. The set $H \subset V$ is an independent set if $x \notin L(H \setminus \{x\}) = < H \setminus \{x\} >, \forall\, x \in H$.

Definition 1.5. An independent generating set $B \subset V$ is called a basis.

Let $\{V, +,*\}$ be a vector space over a field F and let $H \subset V$. The statements of the following theorem are popularly known to be true.

Theorem 1.1

1. A maximal independent set $B \subset V$ is a basis.

2. A minimal generating set $B \subset V$ is a basis i.e. it is independent;

3. If $H \subset V$ is an independent set and $v \in V$ is independent from H then $H \cup \{v\}$ is an independent set. Equivalently any independent set has an extension to a basis.

4. In a vector space $\{V, +,*\}$ over a field F the cardinality of all maximal independent sets is the same, which is the dimension of a vector space.

Proof. We prove them one after the other in the order they are given:

Proof of 1. Readers can also look up detailed explanations and proof in sections 1.3.1 – 1.3.2 of [6]. A slightly similar proof is also given by Sampson et. Al in [2] and [7].

Proof of 2. We show first as given in [2] that a generating set G being minimal $\Rightarrow G$ is independent: Let $x \in G$ be any element. We assume that G is dependent $\Longrightarrow \exists x \in \langle G \backslash \{x\} \rangle$ then

$$(G \setminus \{x\}) \cup \{x\}) \subset \langle G \backslash \{x\} \rangle \Longrightarrow V = \langle G \rangle = \langle (G \backslash \{x\}) \cup \{x\}) \rangle \subset \langle \langle G \setminus \{x\} \rangle \rangle$$

Which contradicts with the minimality of G. Hence G is rather independent, and also being a generating set implies that it is a basis. This completes the proof of 2.

Proof of 3.

See lemma 22 of [6] for the proof of 3.

Remark 1.6.1 (About proof of 4)**.** The proof of 4 is directly implied from the Steinitz Exchange (Replacement) Lemma (see theorem 31 of [6]) about exchange of elements of bases of a set. It states that if there are two bases of a set, then taking one basis, each element of it can be replaced by one element of the other basis. Hence the second basis has at least as many elements as the first one has. Next, the roles of the bases are exchanged. Repeating the process for more bases justifies the equality of the dimension of all the bases of the set.

Example 1.1.1 (Example 2.9 of [2]). Let $\{p_i\}_{i=1}^{m}$, $n = \prod_{i=1}^{m} p_i$ and $\{Z, +\}$ a group. Let

$$G_m := \left\{\frac{n}{p_i} | 1 \le i \le m\right\}$$

Let $\frac{n}{p_i} \in G_m$ be selected:

$$\frac{n}{p_i} = p_1 \times p_2 \times \dots . \times p_{i-1} \times p_{i+1} \times \cdots \times p_m.$$

Since p_i is not in the product, p_i is not its factor.

Consider $G_m \backslash \left\{\frac{n}{p_i}\right\}$. Then $\forall z \in G_m \backslash \left\{\frac{n}{p_i}\right\}$, $p_i / z \Longrightarrow \forall u \in \langle G_m \backslash \left\{\frac{n}{p_i}\right\}\rangle$, $p_i | u$ while $p_i \nmid \frac{n}{p_i}$

In summary, $\frac{n}{p_i} \notin \langle G_m \backslash \left\{\frac{n}{p_i}\right\}\rangle$ meaning that $\left\{\frac{n}{p_i}\right\}$ is independent from $G_m \backslash \left\{\frac{n}{p_i}\right\}$.

Remark 1.1.1 It can be shown with examples (see Example 1.8) that handling generating sets and independence including construction of bases in semigroup theory requires a technique different from the technique used in vector spaces. Important conclusions are the deviations from the properties discussed in Theorem 1.6 about vector spaces stated in theorem 1.10 about semigroups.

1. Independence: There may exist maximal independent set of elements in a semigroup which does not form a generating set.

2. There may exists independent set in a semigroup which cannot be expanded to a generating set.

3. Adding an element independent from an independent set may make the independent set dependent in a semigroup.

1.2 Construction of generating sets and bases in semigroups.

We will first analyze the basis construction in vector spaces.

1.2.1. Basis in Vector Spaces.

The algorithm to select a basis in vector spaces is based on the properties of independence.

Let $\{V, +, *\}$ be a vector space over a field F and let $B \subset V$ hold the selected basis elements, let $B = \emptyset$ for starting. Let $G \subset V$ be the starting system of vectors. Then

(a) If $G \neq \emptyset$ then $\exists x \in G$ otherwise [exit];

(b) If $x \notin L(B)$ then $B = B \cup \{x\}$ and $G = G \setminus L(B)$ otherwise $G = G \setminus \{x\}$.

(c) goto (a).

This algorithm collects a basis in B by the Theorem 1.6 (1) and (3). Since these ((1) and (3)) are not necessarily true in semigroups, this cannot be adopted to semigroups. One most important consideration is that a maximal independent subset of a semigroup may not be a generating set and a basis must be a generating set. Since a generating set may not be reached from an independent set, we will work from starting with a generating set and we will sustain it throughout the process. Note that if $\{S,*\}$ is a semigroup then $S \subset S =< S >$ is a generating set. Therefore $G = S$ always can serve as a starting generating set.

Example 1.8. Consider the following semigroup $\{S,*\}$ where S := {a, b, c}, and the operation

is defined by the operation table (Table 1.):

∗	a	b	c
a	a	c	c
b	c	b	c
c	c	c	c

Table 1: The operation table of {S, ∗}.

Theorem 1.2 The structure {S, ∗} is a semigroup, the operation is associative.

Proof. It suffices to check that associativity holds for ∗ in $(S,*)$:

$a * (b * \mathrm{c}) = (a * \mathrm{b}) * \mathrm{c} = c,$ $\qquad a * (c * \mathrm{b}) = (a * \mathrm{c}) * \mathrm{b} = c,$

$b * (a * c) = (b * a) * c = c,$ $\qquad b * (c * a) = (b * c) * a = c,$

$c * (b * \mathrm{a}) = (c * \mathrm{b}) * \mathrm{a} = c,$ $\qquad c * (a * b) = (c * a) * b = c.$

Clearly $\forall x, y, z \in S, x * (y * \mathrm{z}) = (x * \mathrm{y}) * \mathrm{z}$, so {S, ∗} is a semigroup

Theorem 1.3 In the semigroup $\{S,*\}$ given in Example 1.8, the following are true.

1. The subset $\{b, c\} \subset S$ is a subsemigroup of S. The members of $\{b, c\}$ are independent: $\{b, c\} \setminus \{b\} = \{c\}$ and $b \notin< \{c\} >= \{c\}$. Similarly, $\{b, c\} \setminus \{c\} = \{b\}$ and $c \notin< \{b\} >= \{b\}$.

2. The subset $\{a, b\} \subset S$ is a generating set of S and a and b are independent hence it is a basis and

$a \in S$ is independent from $\{b, c\} \subset S$ since this set is a sub semigroup of S and a is not in it.

3. Since $\{b, c\} \subset S$ is an independent subsemigroup, there is only one element in S which is independent from this set, it is $a \in S$ and adding a to $\{b, c\}$ forms a dependent set:

$$\{a, b, c\} \setminus \{c\} = \{a, b\} \subset S \, gives \, < \{a, b\} >= S \ni c.$$

Proof. We proved the statements of the theorem in the theorem so this does not require a separate proof.

1.2. Remarks. The example shows that handling generating sets and independence including construction of bases in semigroup theory requires a technique different from the technique used in vector spaces. Important conclusions are the deviations from the properties discussed in Theorem 1.6 about vector spaces now stated in theorem 1.0 about semigroups.

1. Independence: There may exist maximal independent set of elements in a semigroup which does not form a generating set.

2. There may exists independent set in a semigroup which cannot be expanded to a generating set.

3. Adding an element independent from an independent set may make the independent set dependent in a semigroup.

2 Bases of Semigroups: The Algorithm.

The following is the first presented algorithmic for obtaining the bases set of any semigroup. We present some below.

Let $\{S,*\}$ be a semigroup and let $\emptyset \neq G \subset S$ be a generating set. Let $S \supset B = \emptyset$.

(a) If $G \neq \emptyset$ then $\exists x \in G$ and $G_{test} = G \setminus \{x\}$ otherwise [exit];

(b) If $x \notin L(G_{test} \cup B)$ then $B = B \cup \{x\} \, and \, G = G \setminus L(B)$ otherwise $G = G_{test}$;

(c) goto (a).

Remark 2.1.

(1). At starting $G \cup B = G$ is a generating set.

(2). In step (a) $G_{test} = G \setminus \{x\}$ removes x from G hence the test $x \notin L(G_{test} \cup B)$ in (b) checks for $x \notin L((G \cup B)\setminus\{x\})$ which checks if x is independent from the generating set $G \cup B$.

Observe that in the case of vector spaces we check for the independence from B only.

(3). In step (b) we change B to a new value $B_{new} = B \cup \{x\}$ and $G_{new} = G \setminus L(B_{new})$ if

the independence of x holds otherwise $G_{new} = G_{test}$ *and* $B_{new} = B$. The set $G_{new} \cup B_{new}$ is

a generating set in the step (c): At step (a) $(G \cup B)$ was a generating set at starting since

$B = \emptyset$ and G was a generating set.

After selecting $x \in G$ we have two options:

i). $x \notin L(G_{test} \cup B)$: Then $x \in B_{new}(= B \cup \{x\})$ and $G_{new} = G_{test}$ hence $x \in G$ moves from

G into $B \subset S$. The final value of G is $G_{new,f} = G_{new} \setminus L(B_{new})$.

The set $G_{new,f} \cup B_{new}$ is a generating set: $L(G_{new,f} \cup B_{new}) \supset G_{new,f} \cup L(B_{new}) \supset G_{new} \cup B_{new}$. Then $G_{new} = G \setminus \{x\}$ and $B_{new} = B \cup \{x\} \Rightarrow G_{new} \cup B_{new} = G \cup B$. Since $G \cup B$ was a generating set and $L(G_{new,f} \cup B_{new}) \supset G \cup B$, $G_{new,f} \cup B_{new}$ is a generating set.

ii). $x \in L(G_{test} \cup B)$: Then $G_{new} = G_{test}$ and $B_{new} = B$.

Since in this case $x \in L(G_{test} \cup B)$ holds, $L(G_{test} \cup B) \supset G_{test} \cup B \cup \{x\} = G \cup B$. Hence $G_{new} \cup B_{new}$ is a generating set.

(4). $G \cup B \supset G_{new,f} \cup B_{new}$ and $G = G_{new,f}$ moreover if $x \in G$ is selected in step (a) then

$x \notin G_{new} \supset G_{new,f}$ hence or otherwise, any x can be selected only once. Therefore, if we select basis vectors $x_1, x_2, \dots, x_k, \dots$ one after the other then

$G_0 := G, B_0 := B = \emptyset,$ $G_1 := G_{new,f}$, $B_1 := B_{new}, \dots,$ $G_k := (G_{k-1})_{new,f}$, $B_k := (B_{k-1})_{new}, \dots$.

Then the system of sets $\{G_S, B_S\}_{1 \le s' < \infty}$ and the selected points $\{x_S\}_{1 \le s \le \infty}$ fulfil:

$$(G_k \cup B_k) \supset (G_{k+1} \cup B_{k+1}); \quad G_k \supset G_{k+1} \textit{ and } G_k \neq G_{k+1} \; \forall \, 0 \le k < \infty$$

$x_j \in B_j \ \forall\, 1 \leq j < k, \ \ B_k = \{x_j \mid 1 \leq j \leq k\}, \ \forall\, k \in N\,.$ (11)

Theorem 2.1.
The set $B_k \subset S$ is independent $\forall 1 \leq k < \infty$.

Proof. The proof is a direct implication of Remarks 2.1 (1) – (4), populating the set B_k is done with independent elements at each stage, giving rise to a set in which each element is not in the subsemigroup generated by the remaining elements up to the last set B_k. Hence B_k is independent.

Theorem 2.2.
In any semigroup $\{S,*\}$ there exists an independent generating set B ⊂ S (basis). If in the semigroup $S\ \exists x, y, z \in S$ such that $x * y = z$ and $z \notin \{x, y\}$, then there exists a basis $B \subset S, B \neq S$ (B is a proper subset of S).

Proof. The existence follows from our reasoning above. If there exists x, y, z ∈ S as stated in the theorem then let us select $z \in G = S$ as the first choice. Then $S\backslash\{z\} \supset \{x, y\} \Rightarrow x * y = z \in \langle\{x, y\}\rangle \subset \langle G\backslash\{z\}\rangle$ hence z is not independent from $G\backslash\{z\}$ hence z is not a selected basis element. Therefore, the selected basis B is a proper subset of S.

3 Discussions and Conclusion

In any semigroup there exists a maximal independent subset that is a generating set hence that is a basis. With the example of the semigroup of integers in [7] and [4], we proved that there are bases in a semigroup which can have different numbers of elements. With the example we showed that in a semigroup there may exists a maximal independent set of elements which is not a generating set.

The idea of the algorithm is as follows: When we select the next candidate to be a basis element from the current generating set then we assure that it is independent from all the previously selected basis elements (this is B) and it is independent from all possible future selections (this is the current G). This puts $x \notin (G \cup B) \setminus \{x\}$ into the selection condition.

4 Recommendation

This work has for now been focused on constructing the algorithm for determining the bases of any semigroup. In a later paper soon, we will present the utilization of the ideas raised here in determining the Basis of some classes of semigroup. This will take us off the need to rely on the popular green's equivalences that have been the tools in several articles over the years. It will be an interesting exercise to compare results starting with bases in Abundant Semigroups, Superabundant semigroup, Orthogroup, the Cyber Group and the Brandt Semigroup as given in [8], [9], [3], [5], [10] and [11].

The characterizations in [11] are used for presenting an algorithm in [7] for generators of the Additive Semigroup of Integers.

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