International Journal of Mathematical Analysis and Modelling

(Formerly Journal of the Nigerian Society for Mathematical Biology)

Volume 7 Issue 1, 2024

ISSN (Print): 2682 - 5694
ISSN (Online): 2682 – 5708
www.tnsmb.org

An Investment Model for a DC Pension System with Refund Clause under Quadratic Utility

K.N.C. Njoku*

Abstract

Following the Nigerian pension reform act of 2004, there is a clause mandating the pension fund administrators (PFAs) in the defined contributory pension scheme to make a refund to their retirement savings account (RSA) holder's family who died during the accumulation phase. As good as this clause is to RSA holders; it posed a serious challenge to PFAs on the investment strategy to be used under this condition. In this paper, the optimal investment strategy (OIS) is studied for defined contributory pension with return of premium clause under quadratic utility function. The charge on balance by pension fund administrators and the mortality risk of members of the scheme during the accumulation period by introducing return of premium clause is considered. To achieve this, a portfolio consisting of a risk free and a risky asset with charge on balance by pension fund administrators is considered and the Abraham De Moivre force function is used to establish the mortality rate of members during accumulation. Furthermore, we obtained an optimization problem using dynamic programming approach. Using Legendre transformation and Dual theory, the value function, OIS with return clause is obtained under quadratic utility function. Finally, numerical simulations of the effect of some sensitive parameters on the OIS were obtained and analyzed.

Keywords and phrases: quadratic utility; optimal investment strategy; return clause of premium; charge on balance; Abraham De Moivre force function; Legendre transformation method.

1 Introduction

In a DC pension scheme, there exist two crucial phases; one the accumulation phase [1-9] followed by the distribution phase [10]. During the accumulation phase, members contribute a certain percentage of their income into their retirement savings account (RSA) which is managed by the pension fund administrator (PFA) by investing the accumulated funds for the purpose of maximizing his members' wealth toward actualizing a better retirement package [11]. During the accumulation phase, members were only supposed to contribute and wait till after retirement for distribution. However, in recent years, this has not been the case since PFAs will have to make sum refunds during the accumulation phase due to some conditions such as loss of job, dead of members or even for the mortgage. This has

*Department of Mathematics, Imo State University, Owerri, Nigeria.

somehow affected the dynamics of the investment strategies already in existence. Hence this has led to the study of optimal investment strategies with return of premium clauses under different assumptions.

In [12], the optimal investment strategy with return of premium clause was studied under mean variance utility function. In their research work, they considered a PFA who invested in one risk free asset and a risky asset where the risky asset followed the geometric Brownian motion (GBM). [13] studied the OIS for a DC pension scheme with return of premium when the risky asset followed the Heston volatility model; they also determine the efficient frontier of the investor. The OIS with return of premium was also studied by [14]. In their work, they considered a portfolio which comprised of one risk free asset and two risky assets bond modeled by geometric Brownian motion (GBM) and stock modeled by constant elasticity of variance and obtained the OIS under mean variance utility. In [15] the authors studied the OIS with return of premium clause where the price process of the risky asset followed the jump diffusion process. [16], investigated investment in one risk free asset, stock and inflation index bond where the stock market price was modeled by Heston's volatility model and proceed to obtain OIS with return clause for both the inflation index bond and the stock market price. The mean variance optimization of portfolio with return of premium clause in a DC plan with multiple contributors under CEV model was studied in [17] by extending the work of [14]. [18], further extended the work of [12] by studying the OIS with return of premium for a portfolio consisting of assets invested in fixed deposit, stock and loan. Furthermore, the explicit solution of an investment strategy with voluntary contribution and return clause was studied under logarithm utility by [19]. However, in all the literature mentioned above, they assumed the returned funds are without interest from the invested funds but in [20-21], they studied the OIS for a DC scheme with return of premium clause where the return of premium was with predetermined interest. In their work, they assumed the refund premiums are with predetermined interest from investment on risk free asset. In each of the above, they used the Abraham De Moivre model as their mortality force function.

Very recently, some authors have modelled their mortality rate using the Weibull force function as their mortality force function and proceed to obtain the OIS for a DC pension scheme using different utility functions.In [22-23], they studied OIS with return clause with proportional administrative charges; in their work, they used the Weibull force function to model the mortality rate of a DC pension scheme and obtained the OIS under exponential and mean variance utilities respectively.

So far in the literature, we observed that several utility functions have been used to determine the OIS with return of premium which include mean variance, exponential, power and logarithm. All through the literature and to the best of my knowledge, the OIS have not yet been studied with return of premium clause using quadratic utility function under any condition. In this paper, we used the quadratic utility in [2] to study the OIS with return of premium and charge on balance. The Abraham De Moivre model is used as the mortality force function to model the death rate of members.

2 Methodology/model formulation

Let $K_t(t)$ represents the price of the risk free asset and its price process is similar to the one in [12] and [20] given by

$$\begin{cases} dK_t(t) = rK_t(t)dt \\ K_0(0) = k_0 > 0 \end{cases}, \tag{2.1}$$

where $r > 0$ is the predetermined interest rate of the risk free asset.

Similarly, the pension fund administrator may also be willing to invest in a risky asset (stock) modeled by the geometric Brownian motion whose price process is given as follows as in [12] and [20]

$$\begin{cases} dL_t(t) = L_t(t)(\mu dt + \sigma dN_t) \\ L_t(0) = 1 \end{cases}, \tag{2.2}$$

where μ is the expected appreciation rate of $L_t(t)$, σ is the volatility of the stock market price and N_t is the Brownian motion generating the available information in the market generated represented by $\mathcal{F}_t$ called the filtration in a complete probability space $(\Omega, \mathcal{F}_t, \mathcal{P})$, where Ω, is a real space and $\mathcal{P}$ a probability measure.

Next, we consider $I(t)$ to be the fraction of the accumulated wealth to be invested in risky asset and $1 - I(t)$, the fraction to be invested in a risk free asset.
Let a be the monthly contributions at a given time by the pension member, u_0 the initial age during the accumulation phase, T the accumulation phase period, and $u_0 + T$ is the terminal age of the member. $W_{\frac{1}{q}, u_0+t}$ is the mortality rate from time t to $t + \frac{1}{q}$, ta is the accumulated contributions at time t, $mtaW_{\frac{1}{q}, u_0+t}$ is the returned contributions to the death members' families within the accumulation period. Also, let α be the charge on balance which is determined based on the value by the pension fund administrators [8].

From the work of [12-23], if we consider the accumulation period $[t, t + \frac{1}{q}]$ and the investment strategy for the risky asset, the differential form of the fund sizeis given as:

$$V\left(t + \frac{1}{q}\right) = \begin{bmatrix} V(t)\left(\left(1 - I(t)\right)\frac{K_{t+\frac{1}{q}}}{K_t} + I(t)\frac{K_{t+\frac{1}{q}}}{K_t} - \alpha\frac{1}{q}\right) \\ +a\frac{1}{q} - mtaW_{\frac{1}{q}, u_0+t} \end{bmatrix}\left(\frac{1}{1 - W_{\frac{1}{q}, u_0+t}}\right) \tag{2.3}$$

$$V\left(t + \frac{1}{z}\right) - V(t) = \begin{bmatrix} V(t)\begin{pmatrix} \left(1 - I(t)\right)\left(\frac{K_{t+\frac{1}{z}}}{K} - \frac{K_t}{K_t}\right) \\ +\pi(t)\left(\frac{L_{t+\frac{1}{z}}}{L_t} - \frac{L_t}{L_t}\right) - \alpha\frac{1}{q} \end{pmatrix} \\ +a\frac{1}{q} - mtaW_{\frac{1}{q}, u_0+t} \end{bmatrix}\left(1 + \frac{W_{\frac{1}{q}, u_0+t}}{1 - W_{\frac{1}{q}, u_0+t}}\right) \tag{2.4}$$

From the work of [12], we have

$$\begin{cases} \frac{1}{q} \to 0, W_{\frac{1}{q}, u_0+t} = A(u_0 + t)dt, \frac{W_{\frac{1}{q}, u_0+t}}{1 - W_{\frac{1}{q}, u_0+t}} = A(u_0 + t)dt, a\frac{1}{q} \to adt, \\ \alpha\frac{1}{q} \to \alpha dt, \frac{K_{t+\frac{1}{q}}}{K} - \frac{K_t}{K_t} \to \frac{dK_t(t)}{K_t(t)}, \frac{L_{t+\frac{1}{q}}}{L_t} - \frac{L_t}{L_t} \to \frac{dL_t(t)}{L_t(t)} \end{cases} \tag{2.5}$$

Substituting (2.5) into (2.4) we have

$$dV(t) = \begin{bmatrix} V(t)\left(I(t)\frac{dL_t(t)}{L_t(t)} + (1-I(t))\frac{dK_t(t)}{K_t(t)} - \alpha dt\right) \\ +a(1-mtA(u_0+t)dt \end{bmatrix}(1 + A(u_0+t)dt) \qquad (2.6)$$

where $A(t)$ is given as the De Moivre force function and u is the maximal age of the life table. According to [25], $A(t)$ and u are related thus

$$A(t) = \frac{1}{u-t} \quad 0 \le t < u$$

This implies that

$$A(u_0+t) = \frac{1}{u-u_0-t} \qquad (2.7)$$

Substituting (2.1), (2.2) and (2.7) into (2.6), we have

$$dV(t) = \begin{bmatrix} \begin{Bmatrix} V(t)\begin{pmatrix} I(t)(\mu - r) + (r-\alpha) \\ +\frac{1}{u-u_0-t} \end{pmatrix} \\ +\left(\frac{u-u_0-(1+m)t}{u-u_0-t}\right)a \end{Bmatrix} dt + V(t)I(t)\sigma dN_t \\ V(0) = v_0 \end{bmatrix} \qquad (2.8)$$

2.1 The Value Function and HJB Equation

Let $I(t)$ be the optimal investment strategy and we define the utility attained by the investor from a given state v at time t as

$$M_{I(t)}(t,v) = E_{I(t)}\left[U(V(T)) \mid V(t) = v\right], \qquad (2.9)$$

where t is the time and v is the wealth. The objective here is to determine the optimal investment strategy and the optimal value function of the investor given as

$$I(t)^* \text{and } M(t,v) = \sup_{I(t)} M_{I(t)}(t,v) \qquad (2.10)$$

Respectively such that

$$M_{I(t)^*}(t,v) = M(t,v). \qquad (2.11)$$

Applying the Ito's lemma and maximum principle, the Hamilton Jacobi Bellman (HJB) equation which is a nonlinear PDE associated with (2.8) is obtained by maximizing $M_{I^*}(t,v)$ subject to the insurer's wealth as follows

$$\begin{pmatrix} M_t + \left(v\left(r - \alpha + \frac{1}{u-u_0-t}\right) + \left(\frac{u-u_0-(1+m)t}{u-u_0-t}\right)a\right)M_v \\ + \sup_{I(t)}\left\{I(t)(\mu - r)M_v + \frac{1}{2}v^2 I^2 \sigma^2 M_{vv}\right\} \end{pmatrix} = 0 \tag{2.12}$$

Differentiating (2.12) with respect to $I(t)$, we obtain the first order maximizing condition for equation (2.12) as

$$I(t)^* = -\frac{(\mu-r)}{v\sigma^2}\frac{M_v}{M_{vv}} \tag{2.13}$$

Substituting (2.13) into (2.12), we have

$$\begin{Bmatrix} M_t + \left(v\left(r - \alpha + \frac{1}{u-u_0-t}\right) + \left(\frac{u-u_0-(1+m)t}{u-u_0-t}\right)a\right)M_v \\ -\frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}\frac{M_v^2}{M_{vv}} \end{Bmatrix} = 0 \tag{2.14}$$

2.2 Legendre Transformation and Dual theory

The differential equation in (2.14) is a nonlinear partial differential equation and is difficult to solve with direct method. In this section, we introduce the Legendre transformation and dual theory it to transform the nonlinear partial differential equation to a linear partial differential equation.

Theorem 2.1: Let $g: \mathcal{R}^n \to \mathcal{R}$ be a convex function for $z > 0$, define the Legendre transform

$$G(z) = \max_v\{g(v) - zv\}, \tag{2.15}$$

where $G(z)$ is the Legendre dual of $g(v)$.

Since $g(v)$ is convex, from theorem 2.1 and [26-27], the Legendre transform for the value function $M(t,v)$ can be defined as follows

$$\widehat{M}(t,z) = \sup\{M(t,v) - zv \mid 0 < v < \infty\} \ 0 < t < T \tag{2.16}$$

where $\widehat{M}$ is the dual of M and $z > 0$ is the dual variable of v.

The value of v where this optimum is achieved is represented by $g(t,z)$, such that

$$g(t,z) = \inf\left\{v \mid M \geq zv + \widehat{M}(t,z)\right\} \ 0 < t < T. \tag{2.17}$$

From (2.17), the function g and $\widehat{M}$ are very much related and can be refers to as the dual of M and are related thus

$$\widehat{M}(t,z) = M(t,g) - zg. \tag{2.18}$$

where

$$g(t,z) = v, M_v = z, \ g = -\widehat{M}_{z.} \tag{2.19}$$

Differentiating (2.18) with respect to $t, andv$

$$\left\{M_t = \widehat{M}_t, \ \ M_v = z, M_{vv} = \frac{-1}{\widehat{M}_{zz}}\right. \tag{2.20}$$

At terminal time T, we define the dual utility in terms of the original utility function $U(v)$ as

$$\widehat{U}(z) = sup\{U(v) - zv \mid 0 < v < \infty\},$$

and

$$G(z) = sup\ \{v \mid U(x) \geq zv + \widehat{U}(z).$$

As a result $\widehat{M}(t,z) = M(t,g) - zg.$

$$G(z) = (U^{'})^{-1}(z), \tag{2.21}$$

Where G is the inverse of the marginal utility U and note that $M(T,v) = U(v)$

At terminal time T, we can define

$$g(T,z) = \inf_{v>0}\left\{v \mid U(v) \geq zv + \widehat{M}(t,z)\right\} and\ \widehat{M}(t,z) = \sup_{v>0}\{U(v) - zv\}$$

so that

$$g(T,z) = (U^{'})^{-1}(z). \tag{2.22}$$

Substituting (2.20) into (2.13) and (2.14), we have

$$\left\{\widehat{M}_t + \begin{bmatrix} g\left(r - \alpha + \frac{1}{u - u_0 - t}\right) \\ + \left(\frac{u - u_0 - (1+m)t}{u - u - t}\right) a \end{bmatrix} z - \frac{1}{2}\left(\frac{(\mu - r)^2}{\sigma^2}\right) z^2 \widehat{M}_{zz}\right\} = 0 \tag{2.23}$$

$$I(t)^* = -\left[\frac{(\mu - r) z \widehat{M}_{zz}}{v\sigma^2}\right] \tag{2.24}$$

From equation (2.19), differentiating (2.23) and (2.24) with respect to z, we have

$$\left\{\begin{matrix} g_t - \begin{bmatrix} g\left(r - \alpha + \frac{1}{u - u_0 - t}\right) \\ + \left(\frac{u - u_0 - (1+m)t}{u - u - t}\right) a \end{bmatrix} - z g_z\left(r - \alpha + \frac{1}{u - u_0 - t}\right) \\ + z g_z \frac{(\mu - r)^2}{\sigma^2} + \frac{1}{2}\left(\frac{(\mu - r)^2}{\sigma^2}\right) z^2 g_{zz} \end{matrix}\right\} = 0 \tag{2.25}$$

$$I(t)^* = -\left[\frac{(\mu - r)}{v\sigma^2} z g_z\right] \tag{2.26}$$

where, $M(T,z) = U(z)$ and $U(z)$ is the marginal utility of the investor. Next, we proceed to solve (2.25) for g considering an investor with quadratic utility, then substitute the solution into (2.26) for the OIS using change of variable approach.

3 Analysis of the model

3.1 Investor's OIS under Quadratic Utility Function

In this section, we follow the work of [2] and define the quadratic utility function as follows

$$U(v) = (v-c)^2, \tag{3.1}$$

where v is the investor's wealth and c is a constant coefficient.

Our interest is to solve (2.25) for g considering a pension fund manager with quadratic utility, then substitute the solution into (2.26) for the OIS using change of variable approach.

From (2.22) and (3.1), the boundary condition is given as

$$g(T,z) = \frac{1}{2}z + c \tag{3.2}$$

Lemma 3.1: The optimal investment plan for a DC plan with refund clause under quadratic utility is by

$$I(t)^* = \frac{c(\mu-r)}{\sigma^2}\left[1 - \frac{1}{v}\left[\left[\frac{u-u_0-T}{u-u_0-t}\right]EXP[(r-\alpha)(t-T)]\right]\right]$$

Proof

Next, we formulate a solution to (2.25) similar to [21] as follows:

$$\begin{cases} g(t,z) = zd(t) + e(t) \\ d(T) = \frac{1}{2},\ e(T) = c, \end{cases} \tag{3.3}$$

Differentiating (3.3), we have

$$\{g_t = zd_t + e_t,\ g_z = d,\quad g_{zz} = 0 \tag{3.4}$$

Substituting (3.4) into (2.25), we have

$$\left\{\begin{matrix} \left[e_t - \left(r - \alpha + \frac{1}{u-u_0-t}\right)e\right] \\ +z\left[d_t - 2\left(r - \alpha + \frac{1}{u-u_0-t}\right)d + \frac{(\mu-r)^2}{\sigma^2}d - \left(\frac{u-u_0-(1+m)t}{u-u_0-t}\right)ad\right] \end{matrix}\right\} = 0 \tag{3.5}$$

Splitting (3.5), we have

$$\begin{cases} e_t - \left(r - \alpha + \frac{1}{u-u_0-t}\right)e = 0 \\ e(T) = c \end{cases} \tag{3.6}$$

$$\begin{cases} d_t - \left[2\left(r - \alpha + \frac{1}{u-u-t}\right) - \frac{(\mu-r)^2}{\sigma^2} - \left(\frac{u-u_0-(1+m)t}{u-u_0-t}\right)a\right]d = 0 \\ d(T) = \frac{1}{2} \end{cases} \tag{3.7}$$

Solving equation (4.6) and (4.7), we obtain

$$d(t) = \frac{1}{2}\left[\frac{u-u_0-T}{u-u_0-t}\right]^{(2+a(1+m)(u-u_0))} EXP\left[\left[2(r-\alpha) - \frac{(\mu-r)^2}{\sigma^2} + (m-1)a\right](t-T)\right] \tag{3.8}$$

$$g(t) = c\left[\left[\left[\frac{u-u_0-T}{u-u_0-t}\right]\right] EXP[(r-\alpha)(t-T)]\right] \tag{3.9}$$

Substituting (3.7) and (3.8) into (3.3) and (3.4), we obtain

$$g(t,z) = \begin{bmatrix} z\frac{1}{2}\left[\frac{u-u_0-T}{u-u_0-t}\right]^{(2+a(1+m)(u-u_0))} EXP\left[\begin{bmatrix} 2(r-\alpha) \\ -\frac{(\mu-r)^2}{\sigma^2} + (m-1)a \end{bmatrix}(t-T)\right] \\ +c\left[\left[\frac{u-u_0-T}{u-u_0-t}\right] EXP[(r-\alpha)(t-T)]\right] \end{bmatrix} \tag{3.10}$$

$$g_z = \frac{1}{2}\left[\frac{u-u_0-T}{u-u_0-t}\right]^{(2+a(1+m)(u-u_0))} EXP\left[\begin{bmatrix} 2(r-\alpha) \\ -\frac{(\mu-r)^2}{\sigma^2} + (m-1)a \end{bmatrix}(t-T)\right]\Bigg\} \tag{3.11}$$

Simplify (4.10) for z, we have

$$z = \begin{bmatrix} 2\left[g - c\left[\left[\frac{u-u_0-T}{u-u_0-t}\right] EXP[(r-\alpha)(t-T)]\right]\right]\left[\frac{u-u_0-T}{u-u_0-t}\right]^{(2+a(1+m)(u-u_0))} \\ \times EXP\left[\left[2(r-\alpha) - \frac{(\mu-r)^2}{\sigma^2} + (m-1)a\right](T-t)\right] \end{bmatrix} \tag{3.12}$$

$g = v$

Substituting (3.10), (3.11) and (3.12) into (2.26), we have

$$I(t)^* = \frac{c(\mu-r)}{\sigma^2}\left[1 - \frac{1}{v}\left[\left[\frac{u-u_0-T}{u-u_0-t}\right] EXP[(r-\alpha)(t-T)]\right]\right], \tag{3.13}$$

which completes the proof

4 Numerical simulations/sensitivity analysis

In this section, some numerical simulations showing the relationship between the OIS and some sensitive parameters are presented and we will discuss the effect of the parameters on OIS as it affects the PFAs behaviour toward investment under quadratic utility function using (4.13). To achieve this, the following data are used similar to [12,24] unless otherwise stated: $\mu = 0.5, c = 0.01, r = 0.3, , v = 1, \alpha = 0.1, \sigma = 0.1, , u = 100, u_0 = 20, T = 20$.

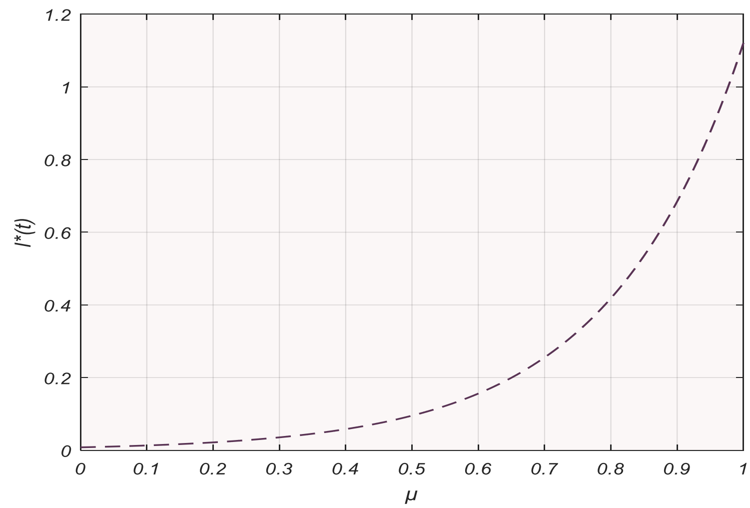

Fig. 1: OIS with appreciation rate of the risky asset

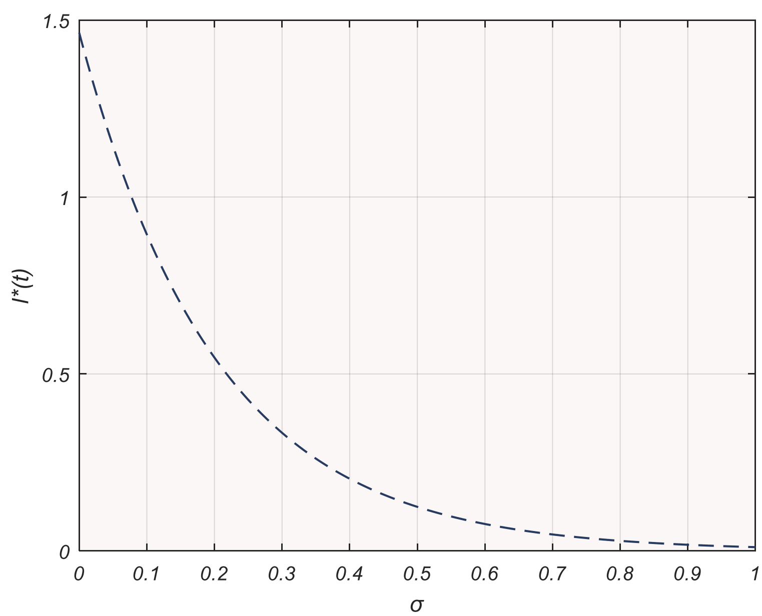

Fig. 2: OIS with instantaneous volatility of the risky asset

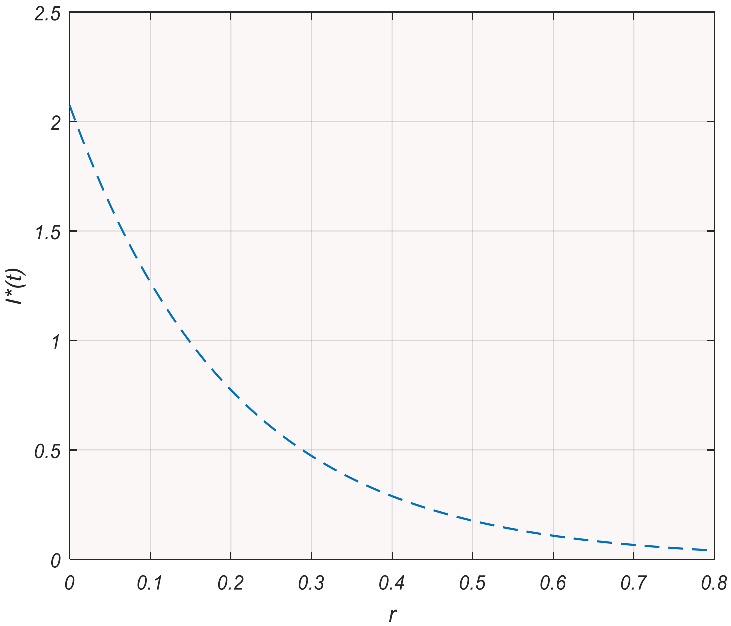

Fig. 3 OIS with predetermined interest rate

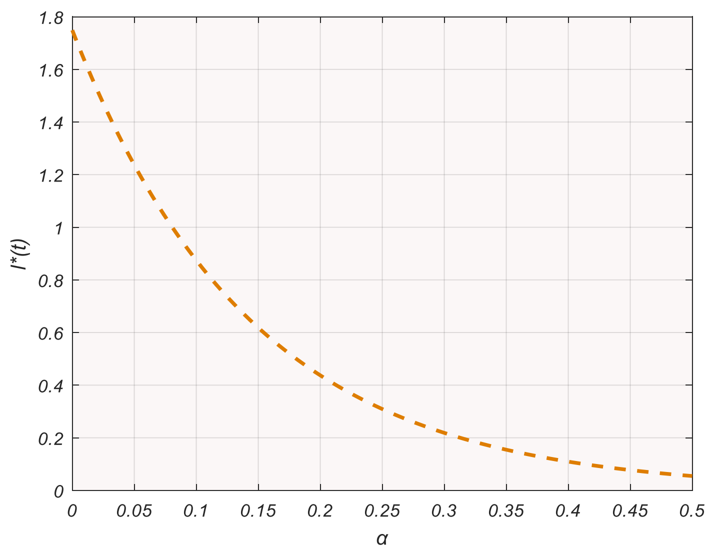

Fig. 4 OIS with charge on balance

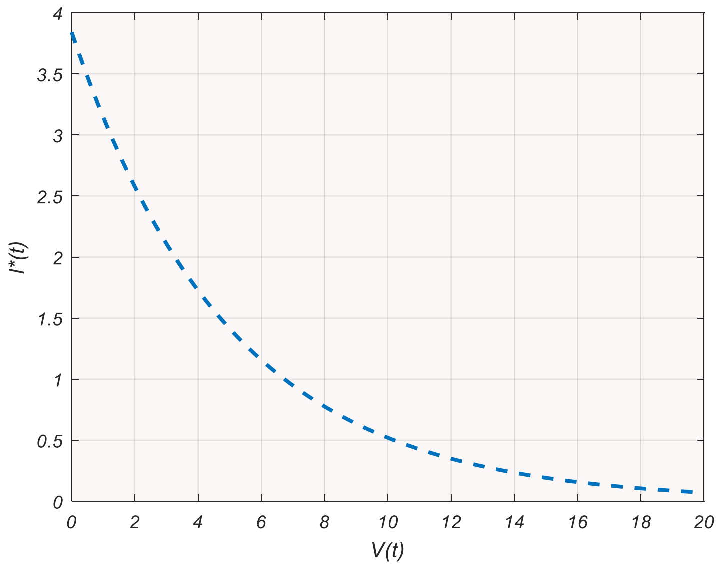

Fig. 5 OIS with wealth of the investor

5 Discussions and conclusion

In fig. 1, the graph of the optimal portfolio distribution against the appreciation rate of the risky asset is presented; the graph shows that the optimal portfolio distribution is an increasing function of the appreciation rate of the risky asset. The implication of the graph is that any asset with higher appreciation rate will naturally be appealing and attractive to an investor; hence the investor may be willing to commit more of his resources into such asset with the expectation of more returns and vice versa.

In fig. 2, the graph of optimal portfolio distribution against the instantaneous volatility is presented; the graph shows that the optimal portfolio distribution is inversely proportional to the instantaneous volatility of the risky asset. This implies that the higher the instantaneous volatility of the risky asset, the higher the risk involved in the investment in such asset. Hence this may create more fears in the mind of the investor toward investing in the risky asset, furthermore reduce the proportion of the investor's wealth in risky asset.

In fig. 3, the graph of the optimal portfolio distribution with the predetermined interest rate is presented; the graph shows that the optimal portfolio distribution is a decreasing function of the predetermined interest rate. The implication of the graph is that a member with large funds prefers to invest where there is lesser risk since they may not want to lose what they have gathered already.

In Fig. 4, the graph of optimal portfolio distribution for the risky asset with tax on invested fund size is presented; the graph shows that, optimal portfolio distribution for the risky asset decreases with increase on tax on the invested fund. The implication of this graph is that members may be discouraged from investment on highly taxed investment and vice versa.

In fig. 5, the graph of optimal portfolio distribution for the risky asset with the investor's wealth is presented. We observed that the optimal portfolio distribution is a decreasing function of the wealth function. This implies that an investor will invest more in the risky assets if their wealth is small and vice versa.

In conclusion, this research work investigated the study of OIS with return of premium clause under quadratic utility function and its application to portfolio optimization and risk management. Also, a portfolio made of a risk free asset and a risky asset which follows the GBM was considered. We took into consideration charge on balance by the PFA. We used the Ito's lemma to obtain an optimization problem in the form of Hamilton Jacobi Bellman equation and investment strategy. These problems are functions of the value function which is dependent on the utility of the investor at the expiration of such investment. Also, the Legendre transformation method and quadratic utility function were used in solving for the OIS. Furthermore, some numerical simulations were also presented to study the effect of some sensitive parameters on the OIS and were observed that OIS under quadratic utility function is an increasing function of appreciation rate of the risky asset, and a decreasing function of charge on balance, instantaneous volatility, member's wealth and predetermined interest rate.

References

[1] Gao, J. (2008). Stochastic optimal control of DC pension funds, Insurance Math. Econom. 42 (3) 1159–1164.

[2] Li, D. Rong, X. and Zhao, H. (2013). Optimal investment problem with taxes, dividends and transaction costs under the constant elasticity of variance model. Transaction on Mathematics, 12(3), 243-255.

[3] Egbe, G. A. Awogbemi, C. A. and Osu, B. O. (2013). Portfolio optimization of pension fund contribution in Nigeria. Mathematical theory and modelling, 3(8), 42-52.

[4] Zhang C and Rong X. Optimal investment strategies for DC pension with astochastic salary under affine interest rate model. Hindawi Publishing Corporation; 2013. Available:http://dx.doi.org/10.1155/2013/297875

[5] Njoku, K.N. C, Osu, B. O., Akpanibah, E. E. and Ujumadu, R. N. (2017). Effect of Extra Contribution on Stochastic Optimal Investment Strategies for DC Pension with Stochastic Salary under the Affine Interest Rate Model. Journal of Mathematical Finance, 7, 821-833.

[6] Akpanibah, E. E. and Samaila**,** S. K. (2017). Stochastic strategies for optimal investment in a defined contribution (DC) pension fund. International Journal of Applied Science and Mathematical Theory, 3(3) 48-55.

[7] Osu, B. O., Akpanibah, E. E. and Njoku, K.N.C. (2017). On the Effect of Stochastic Extra Contribution on Optimal Investment Strategies for Stochastic Salary under the Affine Interest Rate Model in a DC Pension Fund, General Letters in Mathematics, 2(3), 138-149.

[8] Njoku, K. N. C. and Osu, B. O., (2019). On the Modified Optimal Investment Strategy for Annuity Contracts under the Constant elasticity of Variance (CEV) model. *Earthline Journal of Mathematical Sciences*, 1. 169:90.

[9] Njoku, K. N. C., Osu, B. O and Philip, U. U, (2019). On the Investment Approach in a DC Pension Scheme for Default Fund Phase IV under the Constant Elasticity of Variance (CEV) model, *International Journal of Advances in Mathematics*, 2018, 3. 65-76.

[10] Gao J., (2008) Stochastic optimal control of DC pension plan under a CEV model" Insurance: Mathematics and Economics 44, 479-490.
[11] Antolin, P., Payet, S. and Yermo, J. (2010). Accessing default investment strategies in DC pension plan. Financial Market Trends, 2010 (1).
[12] He L. and Liang. Z. (2013). The optimal investment strategy for the DC plan with the return of premiums clauses in a mean-variance framework, *Insurance*, **53**, 643-649.
[13] Sheng, D. and Rong, X. (2014). Optimal time consistent investment strategy for a DC pension with the return of premiums clauses and annuity contracts, Hindawi Publishing Corporation vol**2014**http://dx.doi.org/10.1155/2014/862694. 1-13.
[14] Li, D. Rong, X. Zhao, H. and Yi, B. (2017). Equilibrium investment strategy for DC pension plan with default risk and return of premiums clauses under CEV model, *Insurance:***72,** 6-20.
[15] Chai, Z.P., Rong, X.M. and Zhao, H. (2017). Optimal investment strategy for the DC plan with return of premium. System Engineering- Theory & Practice 37, 1688–1696, (2017).
[16] Wang, Y. Fan, S and Chang, H. (2018). DC Pension Plan with the Return of Premium Clauses under Inflation Risk and Volatility Risk, J. Sys. Sci. & Math. Scis., 38(4), 423-437.
[17] Osu B. O., Akpanibah E. E and Olunkwa O (2018). Mean-Variance Optimization of portfolios with return of premium clauses in a DC pension plan with multiple contributors under constant elasticity of variance model, International journal of mathematical and computational sciences pure and applied researches, 12(5):85-90.
[18] Azor P. A., Avievie I and Esabai M. A. (2023). Explicit solution of an investment plan for a DC pension scheme with voluntary contributions and return clause under logarithm utility. International journal of physical and Mathematical sciences, 17(8), 115-121.
[19] Akpanibah, E. E., Osu, B. O. Oruh, B. I. and Obi, C. N. (2019). Strategic optimal portfolio management for a DC pension scheme with return of premium clauses. *Transaction of the Nigerian association of mathematical physics*, 8(1): 121-130.
[20] Akpanibah E. E. and Osu B. O. (2018). Optimal Portfolio Selection for a Defined Contribution Pension Fund with Return Clauses of Premium with Predetermined Interest Rate under Mean variance Utility. *Asian Journal of Mathematical Sciences.* **2**(2), 19 –29.
[21] Akpanibah, E. E., Osu, B. O. and Ihedioha, S. A. (2020). On the optimal asset allocation strategy for a defined contribution pension system with refund clause of premium with predetermined interest under Heston's volatility model. *J. Nonlinear Sci. A ppl.* **13**(1),:53–64.
[22] Lai C., Liu, S. and Wu, Y. (2021). Optimal portfolio selection for a defined contribution plan under two administrative fees and return of premium clauses. Journal of Computational and Applied Mathematics 398, 1-20.
[23] Njoku K.N. C and Akpanibah, E. E. Modelling and optimization of portfolio in a dc scheme with return of contributions and tax using weibull force function. Asian Journal of Probability and Statistics 16(3): 1-12, 2022.
[24] Chávez-Bedoya, L. (2016). Determining equivalent charges on flow and balance in individual account pension systems, J. Econ. Finance Adm. Sci. 21 (40), 2–7.
[25] Kohler, P.-H. and Kohler, I., (2000). Frailty modeling for adult and old age mortality: the application of a modified De Moivre Hazard function to sex differentials in mortality. Demographic Research 3(8).
[26] Jonsson M. and Sircar, R. (2002). Optimal investment problems and volatility homogenization approximations", Modern Methods in Scientific Computing and Applications 75 255.
[27] Xiao, J. Hong, Z. and Qin, C. (2007). The constant elasticity of variance(CEV) model and the Legendre transform-dual solution for annuity contracts, Insurance, 40(2), 302–31.