International Journal of Mathematical Analysis and Modelling

(Formerly Journal of the Nigerian Society for Mathematical Biology)

Volume 7 Issue 1, 2024

ISSN (Print): 2682 - 5694
ISSN (Online): 2682 – 5708
www.tnsmb.org

The application of Least Squares Method and Shifted Chebychev Polynomials for solving fourth order Volterra Integro-Differential Equations

A.K. Bello[*], A.M. Ayinde[†], A.S. Daodu[‡] and A.A. Ishaq[§]

Abstract

This study employs the least squares method in conjunction with shifted Chebyshev polynomials as a framework to approximate solutions for fourth-order Volterra integro-differential equations. By leveraging this numerical approach, the method effectively tackles various fourth-order Volterra integro-differential equations, yielding results closely aligned with exact solutions. The utilization of the least squares method with shifted Chebyshev polynomials demonstrates its efficacy in solving such equations with exact and converges faster than existing methods. The tables and graphical representation below display this

Keywords and phrases: Chebyshev polynomials; least square method; numerical method; utilized effectively; Volterra integro-differential equations.

1 Introduction

Integro-differential equations, amalgamating integral and derivative operators within an unknown function, find extensive utility across engineering and scientific domains [1]. These equations manifest in forms such as Fredholm, Volterra, or their hybrid, Fredholm-Volterra integro-differential equations, necessitating numerical methods for resolution. Existing literature presents diverse numerical approaches for tackling integro-differential equations, including the collocation method [2-4] and the hybrid linear multistep method [5-7]. Notably, Volterra integro-differential equations, prevalent in realms like fluid mechanics and viscoelasticity, have garnered significant attention [8]. Given the analytical complexity inherent in solving such equations, the pursuit of accurate numerical

[*]Department of Mathematics, University of Ilorin, Ilorin, Nigeria. E-mail: bello.ak@unilorin.edu.ng

[†] Corresponding Author; Department of Mathematics, University of Abuja, Abuja, Nigeria.
E-mail: ayinde.abdullahi@uniabuja.edu.ng

[‡] Department of Mathematics, University of Ilorin, Ilorin, Nigeria. E-mail: samthizidrock@gmail.edu.ng

[§] Department of Physical Sciences, Al-Hikmah University, Ilorin, Nigeria. E-mail: aaishaq@alhikmah.edu.ng

approximations becomes imperative for comparative analysis against exact solutions [9]. Integral equations, particularly Volterra integral equations, constitute a cornerstone in solving myriad natural science predicaments. These equations, encapsulating variable boundary techniques, serve as fundamental tools in modeling numerous phenomena within the natural sciences [10]. The examination of such integral equations has facilitated the translation of various real-world problems into mathematical frameworks, thereby enabling their resolution [10]. Moreover, a pivotal advancement in contemporary mathematics lies in the successful resolution of Volterra integral equations, underscoring their significance in both theoretical and practical domains.

Numerical solutions for various classes of integral equations remain scarce, with only a handful of approximate methods available. Among these methods are several approaches tailored specifically for integral equations with variable boundaries, each presenting its own limitations. In this study, we employ the least squares method in conjunction with shifted Chebyshev polynomials as a basis to approximate solutions for fourth-order Volterra integro-differential equations [11, 12].

An integral equation is an equation in which the unknown function $u(x)$ to be determined appears under the integral sign. A typical form of an integral equation in $u(x)$ is of the form

$$u(x)= f(x)+\int_{\alpha(x)}^{\beta(x)} K(x,t)u(t)dt \tag{1}$$

where $K(x,t)$ is called the kernel of the integral equation, and $\alpha(x), \beta(x)$ are the limits of integration.

In equation (1), it is easily observed that the unknown function $u(x)$ appears under the integral sign as stated above, and out of the integral sign in most other cases [13]. It is important to point out that the kernel $K(x,t)$ and the function $f(x)$ in (1) are given in advance. Our goal is to determine $u(x)$ that will satisfy (1), and this may be achieved by using the Volterra Integro-differential equation

$$u^{(n)}(x)= f(x)+\lambda\int_{0}^{x} K(x,t)u(t)dt \tag{2}$$

This equation (2) is called an integral-differential equation because differential and integral operators are involved in the same equation. The concept of integral and integro-differential equations has inspired a lot of study in recent years [14]. Generally, methods for solving integro-differential equations combine approaches used for both integral and differential equations, which contain both integral and differential operators. Also, since closed-form solutions may not be tractable for most applications, numerical methods are employed to obtain an approximation to exact solutions.

Numerical methods like variational iteration, homotopy pertubation, domain decomposition methods, and others have been used to solve these equations. Other methods, such as least squares methods, where Chebyshev and Legendre's polynomials are used as basis functions, have been applied to obtain solutions to some higher-order linear-type integro-difference equations and their function derivatives [14]. Integro-differential equations emerge frequently as mathematical models in diverse disciplines. The study of integral and integro-differential equations may be traced to the work of Abel, Lotka, Fredholm, Malthus, Verhulst, and Volterra on problems in mechanics, mathematical biology, and economics [15]. The work of Volterra on the problem of competing species is of fundamental importance for the development of mathematical modeling of real-world problems. Volterra explored the effect of inherited characteristics when he was examining a population growth model. The research work led to a specific topic where a distinctive equation emerged, featuring both differential and integral operators [15].

Numerous descriptions of physical occurrences in mathematics contain integro-differential equations. These equations arise in fluid dynamics, biological models, and chemical kinetics. The Adomian decomposition approach for solving boundary value problems for fourth-order integro-differential equations was developed by Hashim [16]. While [17] delved into the numerical solution of Volterra-Fredholm integral equations by moving least squares method and Chebyshev polynomials.

In this study, a method based on the least squares approximation method and the Chebyshev polynomial is introduced to address the following fourth-order integro-differential equation:

$$u^n(x) = f(x) + \lambda \int_0^x k(x,t)u(t)dt \qquad 0 \le t, t \le 1 \tag{3}$$

where $u^n(x) = \frac{d^n u}{dx^n}$ because the resulted equation in (3) combines the differential operator and the integral operator, then it is necessary to define initial conditions $u(0), u'(0), u(0), \cdots, u''(0)$ for the determination of the particular solution $u(x)$ of the Volterra integro-differential equation. The method is proposed by using shifted Chebyshev polynomials for solving the integro-differential equations [18]. One of the most significant advantages of the least square technique is that it improves the accuracy of the model problem predictions. By minimizing the sum of the squared residuals, the method ensures that the model fits the observed data as closely as possible. This approach helps to reduce the errors and uncertainties associated with the model predictions, making it more reliable and robust than the other methods.

1.1 Basic Definitions

This section defines the basic terms that will be frequently used during the research work.

Definition 1: Standard Collocation Method [1]: This is the method partitioning intervals into different points. For example, let say an interval $[a, b]$, N be the number of collocation points to get the desired collocation points, we use;

$x_p = a + \frac{(b-a)}{N} p,$ where $p = 1, 2, \cdots N$.

Definition 2: Power Series Polynomial [1]: A power series is basically an infinite degree polynomial that represents some function. It is generally written in the form ; $y(x) = \sum_{n=0}^{\infty} a_n x^n$, where a_n is the coefficient of the polynomials which are real numbers.

Definition 3: Shifted Chebyshev Polynomials [1]: This is a special type of polynomial generated from Chebyshev polynomial. The shifted Chebyshev has interval $[0, 1]$, while Chebyshev has an interval of $[-1, 1]$.

2 Methodology

In this section, the derivation of the method based on the numerical solution of the Volterra Integro-differential equation using the least squares method and the shifted Chebyshev polynomial of the interval [0, 1] and some examples of the fourth-order Volterra Integro-differential equations will be solved using the shifted Chebyshev polynomial as the basis function [17, 18].

2.1 Construction of Chebyshev Polynomials

The Chebyshev polynomial defined on interval $[-1, 1]$ denoted by $T_n(x)$ is defined as

$$T_n(x) = \cos[n \cos^{-1} x] \tag{4}$$

when $n = 0$

$$T_0(x) = \cos\left[0\cos^{-1} x\right] \tag{5}$$

$\cos 0 = 1$

Thus $T_0(x) = 1$

when $n = 1$

$$T_1(x) = \cos 1 \cos^{-1} x \tag{6}$$

$\cos\left[\cos^{-1} x\right] - x$

Thus $T_1(x) = x$.

Hence, from the recurrence relation, i.e.

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \tag{7}$$

when $n = 1$, we have

$$\begin{aligned} T_2(x) &= 2x(x) - 1 \\ &= 2x^2 - 1 \end{aligned} \tag{8}$$

when $n = 2$, we have

$$\begin{aligned} T_3(x) &= 2x(2x^2 - 1) - T_1(x) \\ &= 4x^3 - 3x \end{aligned} \tag{9}$$

The first few Chebychev polynomial of interval $[-1, 1]$ are as follows

$$\left.\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_2(x) &= 2x^2 - 1 \\ T_3(x) &= 4x^4 - 3x \\ T_4(x) &= 4x^4 - 8x^2 + 1 \\ T_5(x) &= 16x^4 - 20x^3 + 5x \end{aligned}\right\} \tag{10}$$

2.2 Shifted Chebyshev Polynomial on interval $[a, b]$

The nth degree shifted Chebyshev polynomials of the first kind valid in the interval $[a, b]$ is given as

$$T_n^*(x) - \cos\left[n\cos^{-1}\left(\frac{2x - b - a}{b - a}\right)\right], \quad n \geq 0 \text{ and } n \in N \tag{11}$$

and the recurrence relation is given as

$$T_{n+1}^*(x) = 2\left(\frac{2x - b - a}{b - a}\right)T_n^*(x) - T_{n-1}^*(x) \tag{12}$$

To use shifted Chebyshev polynomials on interval $[0, 1]$ we need to change the domain using the following substitution

$$X = 2x - 1, \; 0 \leq x \leq 1 \tag{13}$$

The Chebyshev polynomials valid on $[0, 1]$ denoted by $T_n(x)$ is defined as

$$T_n(x) - \cos\{\mathrm{N}\cos^{-1}(2x - 1)\}, \quad 0 \leq x \leq 1 \tag{14}$$

The recurrence relation is given as

$$T_N(x) = 2(2x - 1)T_N(x) - T_N(x), \quad N \geq 1$$

From equation (11) we obtain
when $N=0$, we obtain

$$T_0(x)=\cos[0\cos^{-1}(2x-1)]=1$$

when $N=1$, we obtain

$$T_1(x)=\cos[\cos^{-1}(2x-1)]=2x-1$$

So the shifted Chebyshev polynomials $T_n(x)$ on $[0,1]$ are obtained as follows

$$\left.\begin{aligned}
&T_0^*(x)=1\\
&T_1^*(x)=2x-1\\
&T_2^*(x)=8x^2-8x+1\\
&T_3^*(x)=32x^3-48x^2+18x-1\\
&T_4^*(x)=128x^4-256x^3+160x^2-32x+1\\
&T_5^*(x)=512x^5-128x^4+1120x^3-400x^2-50x-1\\
&T_6^*(x)=2048x^6-6144x^5+6912x^4-3584x^3+840x^2-72x+1\\
&T_7^*(x)=8192x^7-2867x^6+39936x^5-2688x^4+9408x^3-1560x^2+98x-1
\end{aligned}\right\} \tag{15}$$

2.3 Existence and Uniqueness Solution [19]

In this section, we shall give an existence and Uniqueness solution of equation (3) and its initial conditions.

$$\frac{d^n u}{dx^n}\alpha u+\beta v+\int_\rho^\sigma H(\tau,s)\big(u(s)-v(s)\big)ds \tag{16}$$

$$\frac{d^n v}{dx^n}\varphi u+\xi v+\int_\rho^\sigma Q(\tau,s)\big(u(s)-v(s)\big)ds \tag{17}$$

Here,$u\in D$ and $\in D_1\subseteq R^m$, where each D and D_1 is a closed and bounded domain subset of Euclidean spacesR^m.

Suppose that$\alpha=(\alpha_{ij}),\beta=(\beta_{ij}),\varphi=(\varphi_{ij}),$ and $\xi=(\xi_{ij})$ $(m\times m)$ are non-negative matrices.

The functions

$H(\tau,s)$ and $Q(\tau,s)$ is defined on the domain

$$\left.\begin{aligned}
&(\tau,s,u)\in[0,T]\times[0,T]\times D\\
&(\tau,s,u)\in[0,T]\times[0,T]\times D_1
\end{aligned}\right\} \tag{18}$$

Also, the function $H(\tau,s)$ and $Q(\tau,s)$satisfies the following inequality

$$\|H(\tau,s)\|\le H_1,\ \|Q(\tau,s)\|\le Q_1 \tag{19}$$

$\forall\, u\in[0,T], s\in[0,T]$ and $u,u_1,u_2\in D, v,v_1,v_2\in D_1$ where H_1,Q_1 are positive constants, $0\le s\le\tau\le T$.

We define the non-empty sets as follow

$$D_j=D-\alpha_1\|u_0\|+\beta T\|v_0\|+\alpha_2 T\Lambda_1\|u_0-v_0\| \tag{20}$$

$$D_{1_j}=D_1-\varphi_1\|v_0\|+\xi T\|u_0\|+\varphi_2 T\Lambda_2\|u_0-v_0\| \tag{21}$$

2.3.1 Existence Solution

The study of existence solution of the problem above will be introduced by the following theorem without proof.

Theorem 1

Let the functions $H(\tau,s)$ and $Q(\tau,s)$ be defined in the domain (18), continuous in τ and satisfy the inequalities (19), then

$$u(\tau,u_0)=u_0e^{\alpha\tau}+\int_0^\tau\beta v_0ds+\int_0^\tau e^{\alpha(\tau-s)}\left[\int_\rho^\sigma H(\tau,s)\big(u(s)-v(s)\big)ds\right]$$

$$v(\tau, v_0) = v_0 e^{\varphi\tau} + \int_0^\tau \xi u_0 ds + \int_0^\tau e^{\varphi(\tau-s)} \left[\int_\rho^\sigma Q(\tau,s)\big(u(s)-v(s)\big)ds\right]$$

are the solution of the equations (16) and (17) Source: [19]

2.3.2 Uniqueness Solution

The uniqueness solution of the equation (16) and (17) will be introduced by the following theorem without proof

Theorem 2

Let all assumptions and conditions of the theorem 1 be given then the problem (16) and (17) has a unique solution$u = u_\infty(\tau, u_0, v_0)$, $v = v_\infty(\tau, u_0, v_0)$ on the domain (20) Source: [19]

3 Numerical application and results

Description of the least square method on the general problem considered.

Problem 1

Consider the general fourth order integro- differential equation of the form

$$u^{(iv)}(x) = f(x) + \lambda \int_0^x K(x,t)u(t)dt \tag{22}$$

Together with the initial condition $u(0), u'(0), \cdots u^{n-1}(0)$.

Let the assumed equation be

$$u_n(x) = \sum_{i=0}^{n} a_i T_i^*(x), \quad 0 \le x \le 1 \tag{23}$$

where

$$\sum_{i=0}^{n} a_i T_i^*(x) = a_0 T_0^*(x) + a_1 T_1^*(x) + a_2 T_2^*(x) + a_3 T_3^*(x) + \cdots + a_n T_n^*(x) \tag{24}$$

(24) is modified to satisfy the condition and becomes

$$u_n^*(x) = \sum_{i=4}^{N} a_i T_n^{(iv)}(x) \tag{25}$$

Substituting equation (24) and (25) in (22) and replacing x by t in the integrand

$$\sum_{i=4}^{N} a_i T_n^{(iv)}(x) - f(x) + \lambda \int_0^x k(x,t) \sum_{i=0}^{n} a_i T_i^*(t) dt \tag{26}$$

which could be expressed as

$$\begin{aligned} &a_0 T_0^{(*iv)}(x) + a_1 T_1^{(*iv)}(x) + a_2 T_2^{(*iv)}(x) + a_3 T_3^{(*iv)}(x) + \cdots a_n T_n^{(*iv)}(x) = \\ &f(x) + \lambda \int_0^x \left(a_0 T_0^*(t) + a_1 T_1^*(t) + a_2 T_2^*(t) + a_3 T_3^*(t) + \cdots + a_n T_n^*(t)\right) dt \end{aligned} \tag{27}$$

The residual equation from equation (27) is written as

$$R(a_0, a_1, a_2, a_3, \cdots, a_n) = \left\{ \begin{aligned} &a_0 T_0^{(*iv)}(x) + a_1 T_1^{(*iv)}(x) + a_2 T_2^{(*iv)}(x) + a_3 T_3^{(*iv)}(x) + \cdots a_n T_n^{(*iv)}(x) \\ &- \lambda \int_0^x k(x,t)[a_0 T_0^{(*iv)}(t) + a_1 T_1^{(*iv)}(t) + a_2 T_2^{(*iv)}(t) + \\ &\qquad a_3 T_3^{(*iv)}(t) + \cdots a_n T_n^{(*iv)}(t)]dt - f(x) \end{aligned} \right\} \tag{28}$$

Let

$$S(a_0, a_1, a_2, a_3, \cdots, a_n) = \int_0^1 [R(a_0, a_1, a_2, a_3, \cdots, a_0)]^2 w(x) dx \tag{29}$$

where $S(a_0, a_1, a_2, a_3, \cdots, a_n)$ is to be minimized, $w(x)$ which is the weight function is taken to be 1.

Equation (29) is express as

$$S(a_0, a_1, a_2, a_3, \cdots, a_n) = \int_0^1 \left(a_0 T_0^{(*iv)}(x) + a_1 T_1^{(*iv)}(x) + a_2 T_2^{(*iv)}(x) + a_3 T_3^{(*iv)}(x) + \cdots a_n T_n^{(*iv)}(x)\right) - \lambda \int_0^1 k(x,t)\left[a_0 T_0^{(*iv)}(t) + a_1 T_1^{(*iv)}(t) + a_2 T_2^{(*iv)}(t) + a_3 T_3^{(*iv)}(t) + \cdots a_n T_n^{(*iv)}(t)\right] dt - f(x) \tag{30}$$

Now, S is to minimized and to do this, we set

$$\frac{\delta S}{\delta a_j} = 0, 1, 2, 3, \cdots, n \tag{31}$$

for $j = 0$, we get

$$\frac{\delta S}{\delta a_0} = 2\left[\begin{array}{l}\int_0^1 \left(T_0^{(*iv)}(x) - \lambda \int_a^b k(x,t) T_0^*(t) dt\right) a_0 + \left(T_1^{(*iv)}(x) - \lambda \int_a^b k(x,t) T_1^*(t) dt\right) a_1 \\ + \left(T_2^{(*iv)}(x) - \lambda \int_a^b k(x,t) T_2^*(t) dt\right) a_2 + \left(T_3^{(*iv)}(t) - \lambda \int_a^b k(x,t) T_3^*(t) dt\right) a_3 + \cdots\end{array}\right]$$
$$+ \left(T_n^{(*iv)}(t) - \lambda \int_0^x k(x,t) T_n^*(t) dt\right) a_n - f(x)\left(T_0^{(*iv)}(x) - \lambda \int_0^x k(x,t) T_0^*(t) dt\right) dx = 0 \tag{32}$$

for $j = 1$, we get

$$\frac{\delta S}{\delta a_1} = 2\left[\begin{array}{l}\int_0^1 \left(T_0^{(*iv)}(x) - \lambda \int_a^b k(x,t) T_0^*(t) dt\right) a_0 + \left(T_1^{(*iv)}(x) - \lambda \int_a^b k(x,t) T_1^*(t) dt\right) a_1 \\ + \left(T_2^{(*iv)}(x) - \lambda \int_0^x k(x,t) T_2^*(t) dt\right) a_2 + \left(T_3^{(*iv)}(t) - \lambda \int_a^b k(x,t) T_3^*(t) dt\right) a_3 + \cdots\end{array}\right]$$
$$+ \left(T_n^{(*iv)}(t) - \lambda \int_a^b k(x,t) T_n^*(t) dt\right) a_n - f(x)\left(T_1^{(*iv)}(x) - \lambda \int_0^x k(x,t) T_1(t) dt\right) dx = 0 \tag{33}$$

for $j = 2$, we get

$$\frac{\delta S}{\delta a_2} = 2\left[\begin{array}{l}\int_0^1 \left(T_0^{(*iv)}(x) - \lambda \int_0^x k(x,t) T_0^*(t) dt\right) a_0 + \left(T_1^{(*iv)}(x) - \lambda \int_0^x k(x,t) T_1^*(t) dt\right) a_1 \\ + \left(T_2^{(*iv)}(x) - \lambda \int_a^b k(x,t) T_2^*(t) dt\right) a_2 + \left(T_3^{(*iv)}(t) - \lambda \int_0^x k(x,t) T_3^*(t) dt\right) a_3 + \cdots\end{array}\right]$$
$$+ \left(T_n^{(*iv)}(t) - \lambda \int_0^x k(x,t) T_n^*(t) dt\right) a_n - f(x)\left(T_2^{(*iv)}(x) - \lambda \int_9^x k(x,t) T_2^*(t) dt\right) dx = 0 \tag{34}$$

for $j = 3$, we get

$$\frac{\delta S}{\delta a_3} = 2\left[\begin{array}{l}\int_0^1 \left(T_0^{(*iv)}(x) - \lambda \int_0^x k(x,t) T_0(t) dt\right) a_0 + \left(T_1^{(*iv)}(x) - \lambda \int_0^x k(x,t) T_1(t) dt\right) a_1 \\ + \left(T_2^{(*iv)}(x) - \lambda \int_0^x k(x,t) T_2(t) dt\right) a_2 + \left(T_3^{(*iv)}(x) - \lambda \int_0^x k(x,t) T_3(t) dt\right) a_3 + \cdots\end{array}\right]$$
$$+ \left(T_n^{(iv)}(x) - \lambda \int_0^x k(x,t) T_n(t) dt\right) a_n - f(x)\left(T_3^{(iv)}(x) - \lambda \int_0^x k(x,t) T_3^*(t) dt\right) dx = 0 \tag{35}$$

$\vdots$

for $j = n$, we get

$$\frac{\delta S}{\delta a_n}=2\left[\begin{array}{l}\int_0^1\left(T_0^{(iv)}(x)-\lambda\int_0^x k(x,t)T_0(t)dt\right)a_0+\left(T_1^{(iv)}(x)-\lambda\int_0^x k(x,t)T_1(t)dt\right)a_1\\+\left(T_2^{(iv)}(x)-\lambda\int_0^x k(x,t)T_2(t)dt\right)a_2+\left(T_3^{(iv)}(x)-\lambda\int_0^x k(x,t)T_3(t)dt\right)a_3+\cdots\end{array}\right]$$
$$+\left(T_n^{(iv)}(x)-\lambda\int_0^x k(x,t)T_n(t)dt\right)a_n-f(x)\left(T_n^{(iv)}(x)-\lambda\int_0^x k(x,t)T_n^*(t)dt\right)dx=0 \tag{36}$$

Evaluating the integrals in (32)-(36) gives $(n+1)$ algebraic linear equations $n\times n$ unknown constants $a_0, a_1, a_2, a_3, \cdots, a_n$ which are then substituted back into equation (34) to obtain the required approximate solution.

4 Numerical applications

We have two numerical problems that consist of fourth order linear Volterra integro-differential equation.

Problem 1
The fourth order nonlinear Volterra integro-differential equation of the form

$$u^{(iv)}(x)=24+x+\frac{1}{30}x^6-\int_0^x (x-t)u(t)dt, \tag{37}$$
$$u(0)=0,\ u'(0)=1,\ u''(0)=-1$$

is consider with the exact solution as

$$u(x)=x^4+\sin x \tag{38}$$

Source: [20]
Solution
For $n=7$, equation (34) is given as

$$u(x)=a_0T_0(x)+a_1T_1(x)+a_2T_2(x)+a_3T_3(x)+a_4T_4(x)$$
$$+a_5T_5(x)+a_6T_6(x)+a_7T_7 \tag{39}$$

$$u(x)=a_0+a_1(2x-1)+a_2(8x^2-8x+1)+a_3(32x^3-48x^2+18x-1)+$$
$$a_4(128x^4-256x^3+160x^2-32x+1)+a_5(512x^5-128x^4+1120x^3-400x^2-50x-1)$$
$$+a_6(2048x^6-6144x^5+6912x^4-3584x^3+840x^2-72x+1)$$
$$+a_7(8192x^7-2867x^6+39936x^5-2688x^4+9408x^3-1560x^2+98x-1) \tag{40}$$

Remember that

$$u(0)=0$$
$$0=a_0-a_1+a_2-a_3+a_4-a_5+a_6-a_7 \tag{41}$$
$$a_0=a_1-a_2+a_3-a_4+a_5-a_6+a_7$$

substitute (41) in $u(x)$

$$u(x)=a_1-a_2+a_3-a_4+a_5-a_6+a_7+a_1(2x-1)+a_2(8x^2-8x+1)+a_3(32x^3-48x^2+18x-1)+$$
$$a_4(128x^4-256x^3+160x^2-32x+1)+a_5(512x^5-128x^4+1120x^3-400x^2-50x-1)+$$
$$a_6(2048x^6-6144x^5+6912x^4-3584x^3+840x^2-72x+1)+$$
$$a_7(8192x^7-2867x^6+39936x^5-2688x^4+9408x^3-1560x^2+98x-1) \quad (42)$$

$$u(x)=a_1(2x)+a_2(8x^2-8x)+a_3(32x^3-48x^2+18x-1)+$$
$$a_4(128x^4-256x^3+160x^2-32x)+a_5(512x^5-128x^4+1120x^3-400x^2-50x)+$$
$$a_6(2048x^6-6144x^5+6912x^4-3584x^3+840x^2-72x)+$$
$$a_7(8192x^7-2867x^6+39936x^5-2688x^4+9408x^3-1560x^2+98x) \quad (43)$$

Differentiate equation (43)

$$u'(x)=2a_1+a_2(16x-8)+a_3(96x^2-96x+18)+$$
$$a_4(512x^3+768x^2+320x-32)+a_5(2560x^4-5120x^3+3360x^2-800x+50)+ \quad (44)$$
$$a_6(12288x^5-30720x^4+276648x^3+1680x^2-72)+$$
$$a_7(57344x^6-172032x^5+199680x^4-107520x^3+282246x^2-3136x+98)$$

Recall $u'(x)=1$

$$1=2a_1-8a_2+18a_3-32a_4+50a_5-72a_6+98a_7$$
$$2a_1=1+8a_2-18a_3+32a_4-50a_5+72a_6-98a_7 \quad (45)$$

Substitute (45) in (43)

$$u(x)=[1+8a_1-18a_2+32a_4-50a_5+72a_6-98a_7]x+a_2(8x^2-8x)+a_3(32x^3-48x^2+18x)$$
$$+a_4(128x^4-256x^3+160x^2-32x)+a_5(512x^5-1280x^4+1120x^3-400x^2+50x) \quad (46)$$
$$+a_6(2048x^6-6144x^5+6912x^4-3584x^3+840x^2-72x)+$$
$$a_7(8192x^7-28672x^6+39936x^5-26880x^4+9408x^3-1568x^2+9x)$$

$$u(x)=x+a_2(8x^2)+a_3(32x^3-48x^2)+a_4(128x^4-256x^3+160x^2)$$
$$+a_5(512x^5-1280x^4-1120x^3+400x^2)$$
$$+a_6(2048x^6-6144x^5+6912x^4-3584x^3+840x^2)+ \quad (47)$$
$$+a_6(2048x^6-6144x^5+6912x^4-3584x^3+840x^2)$$
$$+a_7(8192x^7-28672x^6+39936x^5-26880x^4+9408x^3-1568x^2)$$

Differentiating equation (47) once and twice

$$u'(x)=1+a_2(16x)+a_3(96x^2-96x)+a_4(512x^3-768x^2+320x)+a_5(2560x^4$$
$$-5120x^3+3360x^2-800x)+a_6(122888x^5+30720x^4+27648x^3 \quad (48)$$
$$-10752x^2+1680x)+a_7(57344x^6-172032x^5+199680x^4$$
$$-107520x^3+28224x^2-3136x)$$

$$u''(x)=16a_2+a_3(192x^2-96)+a_4(1536x^2-1536x+320)+a_5(10240x^3$$
$$-15360x^2+6720x-800)+a_6(61440x^4-122880x^3+82944x^2-21504x+1680)+$$
$$a_7(344064x^5-860160x^4+798720x^3-322560x^2+56448x-3136) \quad (49)$$

Remember $u''(x)=0$, so that

$$0=16a_2-96a_3+320a_4-800a_5+1680a_6-3136a_7$$
$$16a_2=96a_3-320a_4+800a_5-1680a_6+3136a_7 \quad (50)$$
$$8a_2=48a_3-160a_4+400a_5-840a_6+1568a_7$$

Substitute equation (50) into (47)

$$u(x)=x+a_3\left[32x^2\right]+a_4\left[128x^4-256x^3\right]+a_5\left[512x^5-1280x^4+1120x^3\right]+a_6[2048x^6 -6144x^5+6912x^4-3584x^3]+a_7\left[8192x^7-28672x^6+39936x^5-26880x^4+9408x^3\right] \quad (51)$$

Differentiating equation (50) three times, we have

$$u'(x)=a_3\left[96x^2\right]+a_4\left[512x^3-768x^2\right]+a_5\left[2560x^4-5120x^3+3360x^2\right]+ a_6\left[122888x^5-30720x^4+2764x^3-10752x^2-10752x\right]+ a_7\left[57344x^6-172032x^5+199680x^4-107520x^3+28224x^2\right] \quad (52)$$

$$u''(x)=a_3[192x]+a_4\left[1536x^2-1536x\right]+a_5\left[10240x^3-15360x^2+6720x\right]+ a_6\left[61440x^4-122880x^3+82944x^2-21504x\right]+ a_7\left[344064x^5-860160x^4+798720x^3-322560x^2+56448x\right] \quad (53)$$

$$u'''(x)=192a_3+a_4[3072x-1536x]+a_5\left[30720x^2-30720x+6720\right]+ a_6\left[245760x^3-36864x^2+165888x-21504\right]+ a_7\left[344064x^5-860160x^4+798720x^3-322560x^2+56448x\right] \quad (54)$$

Remember $u'''(x)=-1$

$$-1=192a_3-1536a_4+6720a_5-21504a_6+56448a_7$$
$$192a_3=-1+1536a_4-6720a_5+21504a_6-56448a_7$$
$$32a_3=\frac{-1}{6}+256a_4-1120a_5+3584a_6-9408a_7 \quad (55)$$

Substitute (55) into (44), we have

$$u(x)=x-\frac{x^3}{6}+a_4\left[128x^4\right]+a_5\left[512x^5-1280x^4\right]+a_6\left[737280x^2-737280x+165888\right] +a_7\left[6881280x^3-10321920x^2+4792320x-645120\right] \quad (56)$$

Differentiating four times

$$u^{(iv)}(x)=a_4[3072]+a_5[61440x-30720]+a_6\left[737280x^2-737280x+165888\right] +a_7\left[6881280x^3-10321920x^2+4792320x-645120\right] \quad (57)$$

Replacing x by t (57) we have

$$u(t)=t-\frac{t^3}{6}+a_4\left[128t^4\right]+a_5\left[512t^5-1280t^4\right]+a_6\left[737280t^2-737280t+165888\right] +a_7\left[6881280t^3-10321920t^2+4792320t-645120\right] \quad (58)$$

From equation (31)

$$u^{(iv)}(x)=24+x+\frac{x^6}{30}-\int_0^x (x-1)u(t)dt \quad (59)$$

Substitute equation (59) into (37), which yield

$$u^{(iv)}(x)=24+x+\frac{x^6}{30}-\int_0^x (x-1)\begin{bmatrix} t-\frac{t^3}{6}+a_4\left[128t^4\right]+a_5\left[512t^5-1280t^4\right]+a_6[737280t^2 \\ -737280t+165888]+a_7[6881280t^3-10321920t^2 \\ +4792320t-645120 \\ \end{bmatrix}dt \quad (60)$$

$$u^{(iv)}(x) = 24 + x + \frac{x^6}{30} - \left[\begin{array}{l} \frac{x^3}{6} - \frac{x^5}{120} + a_4\left[\frac{64x^6}{15}\right] + a_5\left[\frac{256x^7}{21} - \frac{128x^6}{3}\right] + a_6\left[\frac{256x^8}{7} - \frac{1024x^7}{7} + \frac{1152x^6}{5}\right] \\ + a_7\left[\frac{1024x^9}{9} - 512x^8 + \frac{6656x^7}{7} - 896x^6\right] \end{array}\right] \tag{61}$$

Equating equation (57) we have our residual equation

$$\begin{aligned} R[a_4, a_5, a_6, a_7] &= -24 - x - \frac{x^6}{30} + a_4\left[\frac{x^3}{6} + 30720\right] + a_5\left[\frac{256x^7}{21} - \frac{128x^6}{3} + 6144x - 30720\right] \\ &+ a_6\left[\frac{256x^8}{7} - \frac{1024x^7}{7} + \frac{1152x^6}{5} + 737280x^2 - 737280x + 165888\right] \\ &+ a_7\left[\frac{1024x^9}{9} - 512x^8 + \frac{6656x^7}{7} - 896x^6 + 6881280x^3 - 10321920x^2 + 4792320x - 645120\right] \end{aligned} \tag{62}$$

Since $S[a_4, a_5, a_6, a_7] = \int_0^1 w(x)R[a_4, a_5, a_6, a_7]^2\,dt$ (63)

We take $w(x) = \left(x - \frac{x^3}{6}\right)$, which satisfy the boundary condition above.

$$S[a_4, a_5, a_6, a_7] = \int_0^1 \left[\begin{array}{l} -24 - x - \frac{x^6}{30} + a_4\left[\frac{x^3}{6} + 30720\right] + a_5\left[\frac{256x^7}{21} - \frac{128x^6}{3} + 6144x - 30720\right] \\ + a_6\left[\frac{256x^8}{7} - \frac{1024x^7}{7} + \frac{1152x^6}{5} + 737280x^2 - 737280x + 165888\right] \\ + a_7\left[\begin{array}{l} \frac{1024x^9}{9} - 512x^8 + \frac{6656x^7}{7} - 896x^6 + 6881280x^3 - 10321920x^2 \\ + 4792320x - 645120 \end{array}\right] \end{array}\right]\left(x - \frac{x^3}{6}\right)dx \tag{64}$$

Finding the values of $a'_j\ s = 4, 5, 6, 7$ which minimizes S is equivalent to finding the best approximation for (28). The minimized value of S is obtained by setting
$\frac{\partial S}{\partial a_j} = 0$, where $j = 4, 5, 6, 7$.

For $j = 4$, we have

$$\frac{\partial S}{\partial a_4} = \frac{3859046311}{111375} \tag{65}$$

For $j = 5$, we have

$$\frac{\partial S}{\partial a_5} = \frac{401968793539}{3711015} \tag{66}$$

For $j = 6$, we have

$$\frac{\partial S}{\partial a_6} = \frac{558308474192926}{1206079875} \tag{67}$$

For $j = 7$, we have

$$\frac{\partial S}{\partial a_7} = \frac{62085449747343752}{41247931725} \tag{68}$$

The various values obtained in solving the equations (65)-(68) above using gaussian elimination are as follows

$$\left.\begin{aligned} a_4 &= 0.007966823275 \\ a_5 &= 0.0000141524812 \\ a_6 &= -3.20592\times10^{-7} \\ a_6 &= -2.03596\times10^{-8} \end{aligned}\right\} \tag{69}$$

While the values of $a_1, a_2, a_3 = 0.862723069127166, 0.1894068028279015, 0.0580009954279908$

The values of $a_1, a_2, a_3, a_4, a_5, a_6, a_7$ are substituted into $u(x)$ to get

$$\begin{aligned} u(x) &= x - 5.08273978461264\times10^{-16} x^2 - 0.16666666666666665 x^3 + 0.99996953698625 x^4 \\ &+ 0.008402706266117185 x^5 - 0.00007282211644465463 x^6 - 0.0001667862003381444 x^7 \end{aligned} \tag{70}$$

4.1 Tables

Table 1. The numerical result of problem 1

x	Exact solution	Approximate Solution	Absolute Error
0.1	-0.8951707486	-0.8951707494	2.50e-09
0.2	-0.7813972528	-0.7813972471	3.09e-08
0.3	-0.6598162090	-0.6598162825	1.24e-07
0.4	-0.5316426245	-0.5316426517	3.17e-07
0.5	-0.3981568190	-0.3981570232	6.32e-07
0.6	-0.2606922898	-0.2606931415	1.67e-06
0.7	-0.1206216388	-0.1206245001	1.73e-06
0.8	0.0206576891	0.0206493816	2.57e-06
0.9	0.1617383810	0.1617169414	3.66e-06
1.0	0.3012188203	0.3011686789	5.02e-06

Problem 2

Consider the fourth order Volterra-integro differential equation

$$u^{(iv)}(x) = -1 + x - \int_0^x (x-t)u(t)\,dt \tag{71}$$

with initial condition as $u(0)=-1, u'(0)=1, u''(0)=1, u'''(0)=-1$ well it's the exact solution as

$$u(x) = \sin x - \cos x \tag{72}$$

Source: [20].

Table 2: the numerical result of problem 2

x	Exact Solution	Computed Solution	Absolute Error
0.1	-0.8951707486	- 0.8951707494	2.546e-08
0.2	-0.7813972528	- 0.78139724705	2.177e-07
0.3	-0.6598162090	- 0.6598162825	7.350e-08
0.4	-0.5316426245	- 0.5316426517	2.720e-08
0.5	-0.3981568190	- 0.3981570232	2.041e-07
0.6	-0.2606922898	- 0.2606931415	8.517e-07
0.7	-0.1206216388	- 0.1206245001	2.940e-06
0.8	0.0206576891	0.0206493816	8.310e-06
0.9	0.1617383810	0.1617169414	2.144e-05
1.0	0.3012188203	0.3011686789	5.014e-05

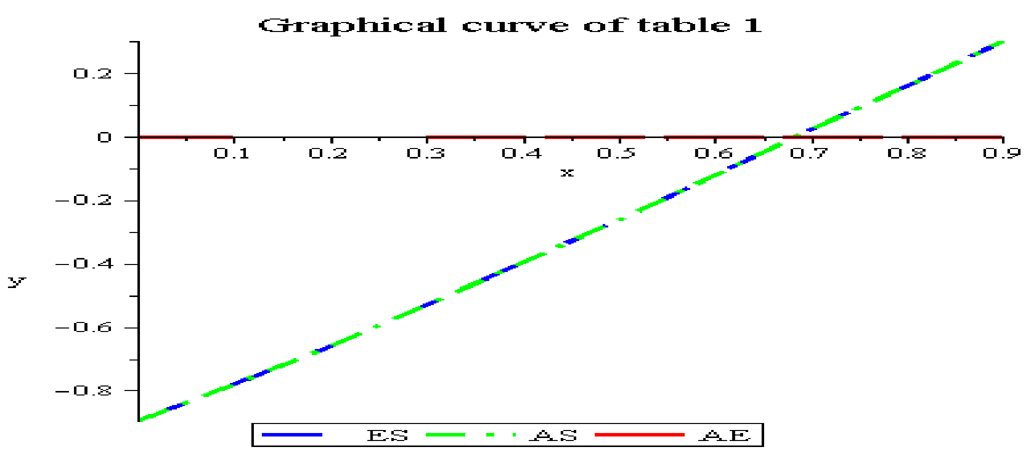

Fig. 1. Graphical Illustration of problem 1

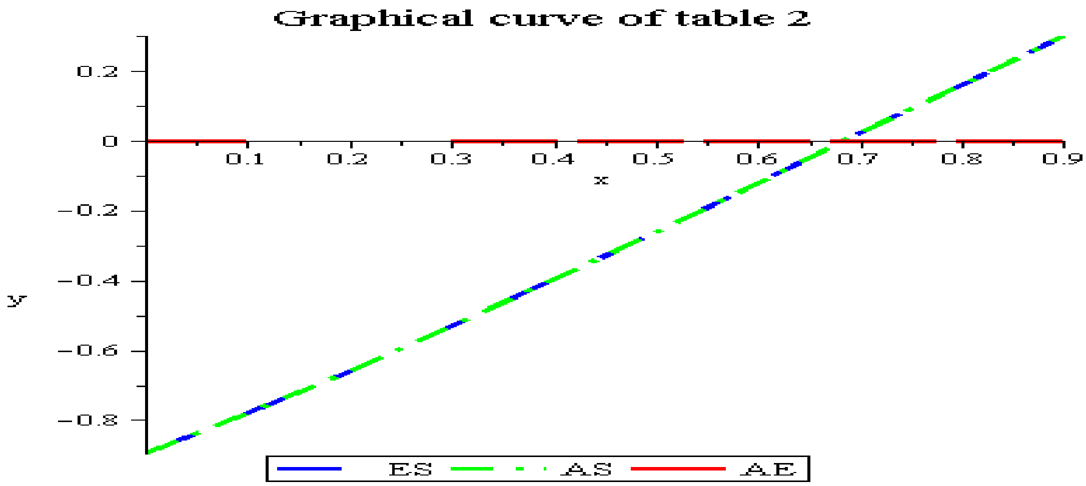

Fig. 2. Graphical Illustration of problem 2

5. Discussion and conclusion

The study focuses on the numerical solution of fourth-order Volterra integro-differential equations using the Least Squares Method and shifted Chebyshev polynomials as the basis function. The numerical method of solving and some numerical problems on fourth order Volterra integro-differential equation using the Least squares method and shifted Chebychev polynomials as basis function were consider. Table 1 and 2 shows the numerical results of problem 1 and 2, while figure 1 and 2 shows the curve of table 1 and 2. The computation of results obtained from the application of the method shows that, the Least Square method using shifted Chebyshev Polynomial as basis function has been utilized effectively to solve the fourth order Volterra Integro-Differential Equations. The solution obtained through this method produce results very close to the exact solution compare to [1]. The simplicity of the method is an included advantage and thus, it is dependable and a power tool for solving the class of problem considered. Conclusively, the study considered the solution of Volterra integro-differential equations of fourth order. All computations are done with the aid of **Maple 18** software.

References

[1] Ayinde, M.A., Ishaq, A.A., Latunde, T. and Sabo, J. (2021). Efficient numerical approximation methods for solving high-order integro-differential equations. Caliphate Journal of Science & Technology. 2, 188-195.

[2] Ajileye, G. and Aminu F.A. (2022). Approximate Solution to First-Order Integro-differential Equations Using Polynomial Collocation Approach. J. Appl Computat Math. 11, 1-4.

[3] Shahooth, M.K., Ahmad, R., Salman, U. and Swidan, W. (2016). Approximation solution of solving linear Volterra Fredholm integro differential equations of the second kind by using Bernstein

polynomial method. J. Appl. Computat. Math. 5(2), 298. doi:10.4172/2168-9679.1000298
[4] Ajileye G., Oyedepo T., Adiku Lydia, and Sabo, J. (2023). Collocation based numerical method for multi-order fractional integro-diffeerential equation. Iranian Journal of Numerical Analysis and Optimization. 13(4), 604-626.
[5] Ajileye, G., Amoo, S.A. and Ogwumu, O.D. (2018). Hybrid block method algorithms for solution of first order initial value problems in ordinary differential equations. J. Appl. Computat. Math. 7:2. doi: 10.4172/2168-9679.1000390.
[6] Ajileye, G., Okedayo, G.T. and Aboiyar, T. (2017). Laguerre collocation approach for continuous hybrid block method for solving initial value problems of second order ordinary differential equations. Trends Food Sci. Techn. 2, 379- 383.
[7] Al-Bugami, A.M. and Al-Juadi, J.G. (2020). Runge-Kutta method and block by block method to solve nonlinear fedholm-volterra integral equation with continuous kernel. Journal of Applied Mathematics and Physics, 8(9), 20-43.
[8] Shoukralla, E.S. and Ahmed, B.M. (2016). Numerical solutions of Volterra integral equations of the second using lagrange interpolation via thee vandermonde matrix. Int. Journal of Physics: Conference eries. IOP publish, 1447(1), 12003.
[9] Adesanya, A.O. Yahaya, Y.A. Ahmed, B. and Onsachi, R.O. (2019). Numerical solution of linear integral and integro-differential equations using boubakar collocation method. International Journal of Mathematical Analysis and Optimization: Theory and Application, 2, 592-598.
[10] Aggarwal, S., Sharma, N. and Chauhan, R. (2018). Solution of linear Volterra integral equations of second kind using mohand transform. International Journal of Research and Advance Technology, 6, 3098-3102.
[11] Abubakar, A. and Taiwo O.A. (2014). Integral collocation approximation methods for the numerical solution of high-orders linear Fredholm-Volterra integro-differential equations. J. Comput. Appl. Math. 4, 111-117.
[12] Ajileye, G. James, A.A. Ayinde, A.M. Oyedepo, T. (2022). Collocation approach for the computational solution of Fredholm-Volterra fractional order of integro-differential equations. J. Nig. Soc. Phys. Sci. 4, 834.
[12] Butris, R.N. and Noori, N.R. (2022). A study in the existence and uniqueness solution of Volterra-Fredholm Integro-differential equations. Journal of Applied Computer Science and Mathematics. 16(2), 26-30.
[13] Bôcher, M. (1926). An introduction to the study of integral equations. University Press. 10.
[14] Wazwaz, A.M. (2015). A first course in integral equations. Second edition. Saint Xavier University, USA.
[15] Volterra, V. (1860-1940). Obituary Notices of fellows of the royal society, 3, (1941): 690-729.
[16] Hashim, I. (2006). Adomian decomposition method for solving BVPs for fourth-order integro-differential equations. Journal of Computational and Applied Mathematics. 193(2), 658-664.
[17] Dastjerdi, H.L., and Ghaini, F.M. (2012). Numerical solution of Volterra-Fredholm integral equations by moving least square method and Chebyshev polynomials. Applied Mathematical Modelling. 36 (7), 3283-3288.

[18] Kythe, P. and Puri, P. (2002). Computational methods for linear integral equations. Springer Science & Business Media.
[19] Wazwaz, A.M. (2011). Linear and nonlinear integral equations: methods and applications. Springer.