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Uniqueness of solution of multi-term fractional order Volterra Integro-differential equations with convergence analysis

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Abstract

This paper focused on multi-term fractional order Volterra integro-differential equations. The multi-term fractional order derivative part of the multi-term fractional order Volterra integro-differential equations is converted into its equivalent integral equation then Banach contraction principle is utilised in the establishment of the uniqueness of solution for multi-term fractional order Volterra integro-differential equations under some condition. Furthermore, the convergence of the solution is also studied and proved. Some examples were considered to test the applicability of the proposed uniqueness theorem for the solution of multi-term fractional order Volterra integro-differential equations.

Keywords and phrases: Uniqueness of solution; fractional integro-differential equation; Convergence; Volterra integro-differential equation

1 Introduction

In the past few decades, Fractional calculus have attracted so much attention of scholars in various field of mathematics and sciences at large. In recent times, numerous scholars have studied the existence and uniqueness of solutions of these equations such as [1] proved the existence results for fractional integro-differential equations with non-local condition via resolvent operators. Also, the existence of solutions of various types of fractional differential equations and fractional integro-differential equations of boundary value problems and initial value problems were explored recently by [2 - 20] and so many more. Furthermore, the application of these equations has been recorded in many discipline of engineering, social sciences and management. This is to say, fractional calculus is widely applied to described many important phenomena in the field of sciences such as Chaotic systems [21], Chemical Reactions [22], Biology [23], Fluid Mechanics [24]. In Economics [25], Social Sciences [26], Finance [27] and so on. Many physical phenomena are more favourably described by fractional operators because it considers the evolution of these phenomena into account. However, it is sometimes difficult to find analytical solutions for some of these equations. Therefore, the need for a numerical approach. In the past few decades, numerical approaches to find approximations to the solutions to this class of equations were studied. See [18, 28 - 33]. A special class of these equations are the Volterra type, it has been used to describe heat transfer issues, nano hydrodynamics, mass diffusion processes, neutron diffusion, biological species coexisting together with diminishing and increasing rate of growth, and electromagnetic theory [34]

In this paper, we considered the multi-term fractional order Volterra integro-differential equation of the form

$$D^{\beta}\xi\left(x\right)=\sum_{j=0}^{k}c_{j}(x)D^{\delta_{j}}\xi\left(x\right)+h\left(x\right)+\int_{0}^{\sigma}K\left(\sigma,\tau\right)H\left(\xi\left(\tau\right)\right)d\tau \tag{1}$$

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subject to the initial condition

$$\begin{aligned} \xi^{(n)}(0) &= d_n,\ n = 0, 1, 2, ..., m-1, \\ m-1 &< \delta_0 < \delta_1 < \cdots < \delta_j < \beta \leq m,\ m, k \in \mathbb{N}, \end{aligned} \tag{2}$$

where D is the differential operator defined in Caputo sense, $\xi : Q \longrightarrow \mathbb{R},\ Q = [0, 1]$ is a continuous function which needs to be determined, $c_j, h : Q \longrightarrow \mathbb{R}$ are given continuous functions, $K : Q \times Q \longrightarrow \mathbb{R}$ is the kernel of integration which is also continuous, $H : \mathbb{R} \longrightarrow \mathbb{R}$ is a Lipschitz function.

2 Peliminaries

Definition 2.1. *(Contraction Map [35]). Let X be a metric space, a mapping $T : X \longrightarrow X$ is a contraction if there exists $L \in [0, 1)$ such that $\|Tx - Ty\| \leq L\|x - y\|$ for all $x, y \in X$.*

Definition 2.2. ***(Fixed Point** [36]). Given a map $T : A \longrightarrow B$, every solution y of the equation*

$$Ty = y$$

is called a fixed point of T.

Definition 2.3. (***Banach Contraction Principle*** [37]). *Let X be a complete metric space, then each contraction mapping $T : Y \longrightarrow Y$ has a unique fixed point y of T in Y; that is, $Ty = y$.*

Definition 2.4. ***(Riemann-Liouville Fractional Integral** [38]). Reimann-Liouville fractional integral of order β of a function ξ is defined as*

$$I^{\beta}\xi(x) = \frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}\xi(s)\,ds, \qquad x > 0,\ \beta \in \mathbb{R}^+, \tag{3}$$

where $\mathbb{R}^+$is the set of positive real numbers.

Definition 2.5. ***(Riemann-Liouville Fractional Derivative** [38]). Reimann-Liouville fractional derivative of order β of a function ξ is defined as*

$$\begin{aligned} D^{\beta}\xi(x) &= D^m I^{m-\beta}\xi(x), \qquad m-1 < \beta \leq m,\ m \in \mathbb{N} \\ &= \frac{d^m}{dt^m}\left(\frac{1}{\Gamma(m-\beta)}\int_0^x (x-s)^{m-\beta-1}\xi(s)\,ds\right). \end{aligned}$$

Definition 2.6. ***(Caputo Fractional Derivative** [38]). The fractional derivative of $\xi(x)$ in the Caputo sense is defined by*

$$\begin{aligned} D^{\beta}\xi(x) &= I^{m-\beta}D^m\xi(x) \\ &= \frac{1}{\Gamma(m-\beta)}\int_0^x (x-s)^{m-\beta-1}\frac{d^m\xi(s)}{ds^m}ds, \quad m-1 < \beta \leq m. \end{aligned} \tag{4}$$

With the following properties

i. $I^{\beta}D^{\beta}\xi(x) = \xi(x) - \sum_{n=0}^{m-1}\frac{\xi^{(n)}(0)}{n!}x^n$, $m-1 < \beta \leq m$,

ii. $I^{\beta}D^{\gamma}\xi(x) = I^{\beta-\gamma}\xi(x)$, $0 < \gamma < \beta$, *and* $m-1 < \beta \leq m, m \in \mathbb{N}$,

iii. $I^{\alpha}\xi(x) = \frac{x^{\alpha}}{\Gamma(\alpha+1)}$, *where* $y(x) = 1,\ x \in [0, 1]$.

3 Main Result

In this work, we denote by

(i) $\|\cdot\|_\infty$ the sup norm on $C(Q,\mathbb{R})$, i.e for $c \in C(Q,\mathbb{R})$, $\|c\|_\infty = \sup_{x\in Q}|c(x)|$.

(ii) $\|\cdot\|_\infty$ the sup norm on $C(Q,\mathbb{R})$, i.e for $h \in C(Q,\mathbb{R})$, $\|h\|_\infty = \sup_{x\in Q}|h(x)|$.

We make the following hypotheses:

(p_1) there exists a constant $\Psi > 0$ such that for any $\xi_1, \xi_2 \in C(Q,\mathbb{R})$ we have

$$|H(\xi_1(x)) - H(\xi_2(x))| \le \Psi \|\xi_1 - \xi_2\|_\infty \quad x \in Q$$

(p_2) there exists a constant Π such that

$$\Pi = \sup_{\sigma\in[0,1]} \int_0^\sigma |K(\sigma,\tau)|\, d\tau < \infty.$$

Lemma 1. Let $\xi : Q \longrightarrow \mathbb{R}$ and $g : Q \longrightarrow \mathbb{R}$ be continuous functions. Then, a function ξ is a solution to the fractional integro-differential equation (1) − (2) if, and only if,

$$\xi(x) = \sum_{n=0}^{m-1} \frac{d_n}{n!}x^n + \sum_{j=0}^{k} c_j(x) I^{\beta-\delta_j}\xi(s) + I^\beta h(x) + I^\beta\left(\int_0^\sigma K(\sigma,\tau) H(\xi(\tau))\, d\tau\right). \quad (5)$$

Proof. Applying equation (3) on equation (1) and using property (i), (ii) together with the initial condition (2) we have,

$$\begin{aligned}
I^\beta\left(D^\beta\xi(x)\right) &= I^\beta\left(\sum_{j=0}^{k} c_j(x) D^{\delta_j}\xi(x)\right) + I^\beta(h(x)) + I^\beta\left(\int_0^\sigma K(\sigma,\tau) H(\xi(\tau))\, d\tau\right) \\
&= \sum_{j=0}^{k} c_j(x) I^\beta\left(D^{\delta_j}\xi(x)\right) + I^\beta(h(x)) + I^\beta\left(\int_0^\sigma K(\sigma,\tau) H(\xi(\tau))\, d\tau\right) \\
&= \sum_{n=0}^{m-1} \frac{\xi^{(n)}(0)}{n!}x^n + \sum_{j=0}^{k} c_j(x) I^{\beta-\delta_j}\xi(x) + I^\beta g(x) + \\
&\quad I^\beta\left(\int_0^\sigma K(\sigma,\tau) H(\xi(\tau))\, d\tau\right) \\
&= \sum_{n=0}^{m-1} \frac{d_n}{n!}x^n + \sum_{j=0}^{k} c_j(x) I^{\beta-\delta_j}\xi(x) + I^\beta h(x) + I^\beta\left(\int_0^\sigma K(\sigma,\tau) H(\xi(\tau))\, d\tau\right).
\end{aligned}$$

Thus, ξ solves (1) − (2) if, and only if, ξ solves (5)

Theorem 3.1. *(Uniqueness of Solution) Assume that* (p_1) *and* (p_2) *holds, and if,*

$$\left(\sum_{j=0}^{k} \frac{\|c_j\|_\infty}{\Gamma(\beta-\delta_j+1)} + \frac{\Pi\Psi}{\Gamma(\beta+1)}\right) < 1, \quad (6)$$

then there is a unique solution $\xi \in C(Q,\mathbb{R})$ *to problem* (1) − (2).

Proof. Let T be an operator such that $T : C(Q,\mathbb{R}) \longrightarrow C(Q,\mathbb{R})$ defined from equation (5) as

$$\begin{aligned}(T\xi)(x) &= \sum_{n=0}^{m-1}\frac{d_n}{n!}x^n + \sum_{j=0}^{k}\frac{1}{\Gamma(\beta-\delta_j)}\int_0^x (x-\sigma)^{\beta-\delta_j-1}c_j(\sigma)\xi(\sigma)d\sigma \\ &+\frac{1}{\Gamma(\beta)}\int_0^x (x-\sigma)^{\beta-1} g(\sigma)\,d\sigma \\ &+\frac{1}{\Gamma(\beta)}\int_0^x (x-\sigma)^{\beta-1}\left(\int_0^\sigma K(\sigma,\tau)H(\xi(\tau))\,d\tau\right)d\sigma. \qquad (7)\end{aligned}$$

The objective here is to apply Banach contraction principle. To do that, we will show that T is a contraction.

First, we note that T is well defined. Indeed, since $x \longmapsto \sum_{n=0}^{m-1}\frac{d_n}{n!}x^n$, $x \longmapsto \sum_{j=0}^{k}c_j(x)\left(I^{\beta-\delta_j}\xi\right)(x)$, $x \longmapsto \left(I^{\beta}h\right)(x)$, $x \longmapsto \int_0^\sigma K(\sigma,\tau)H(\xi(\tau))\,d\tau$ are continuous, the right hand of equation (8) is well defined and $x \longmapsto (T\xi)(x)$ is continuous. Thus, for $\xi \in C(Q,\mathbb{R})$, $T\xi$ is also in $C(Q\mathbb{R})$.

Let $\xi_1, \xi_2 \in C(Q,\mathbb{R})$ and let $x \in [0,1]$. By definition of T we have

$$\begin{aligned}\left|(T\xi_1)(x)-(T\xi_2)(x)\right| &= \left|\sum_{j=0}^{k}\frac{1}{\Gamma(\beta-\delta_j)}\int_0^x (x-\sigma)^{\beta-\delta_j-1}c_j(\sigma)\left(\xi_1(\sigma)-\xi_2(\sigma)\right)d\sigma\right. \\ &\left.+\frac{1}{\Gamma(\beta)}\int_0^x (x-\sigma)^{\beta-1}\left(\int_0^\sigma K(\sigma,\tau)\left(H(\xi_1(\tau))-H(\xi_2(\tau))\right)d\tau\right)d\sigma\right| \\ &\le \left|\sum_{j=0}^{k}\frac{1}{\Gamma(\beta-\delta_j)}\int_0^x (x-\sigma)^{\beta-\delta_j-1}c_j(\sigma)\left(\xi_1(\sigma)-\xi_2(\sigma)\right)d\sigma\right| \\ &+\left|\frac{1}{\Gamma(\beta)}\int_0^x (x-\sigma)^{\beta-1}\left(\int_0^\sigma K(\sigma,\tau)\left(H(\xi_1(\tau))-H(\xi_2(\tau))\right)d\tau\right)d\sigma\right| \\ &\le \sum_{j=0}^{k}\frac{1}{\Gamma(\beta-\delta_j)}\int_0^x (x-\sigma)^{\beta-\delta_j-1}\left|c_j(\sigma)\right|\left|\xi_1(\sigma)-\xi_2(\sigma)\right|d\sigma \\ &+\frac{1}{\Gamma(\beta)}\int_0^x (x-\sigma)^{\beta-1}\left(\int_0^\sigma \left|K(\sigma,\tau)\right|\left|H(\xi_1(\tau))-H(\xi_2(\tau))\right|d\tau\right)d\sigma \\ &\le \sum_{j=0}^{k}\frac{\|c_j\|_\infty\|\xi_1-\xi_2\|_\infty}{\Gamma(\beta-\delta_j)}\int_0^x (x-\sigma)^{\beta-\delta_j-1}d\sigma \\ &+\frac{\Pi\Psi\|\xi_1-\xi_2\|_\infty}{\Gamma(\beta)}\int_0^x (x-\sigma)^{\beta-1}d\sigma \\ &\le \sum_{j=0}^{k}\frac{\|c_j\|_\infty\|\xi_1-\xi_2\|_\infty}{\Gamma(\beta-\delta_j+1)}x^{\beta-\delta_j}+\frac{\Pi\Psi\|\xi_1-\xi_2\|_\infty}{\Gamma(\beta+1)}x^{\beta}\end{aligned}$$

Thus,

$$\|T\xi_1-T\xi_2\|_\infty \le \left(\sum_{j=0}^{k}\frac{\|c_j\|_\infty}{\Gamma(\beta-\delta_j+1)}+\frac{\Pi\Psi}{\Gamma(\beta+1)}\right)\|\xi_1-\xi_2\|_\infty, \text{ for all } x\in[0,1].$$

We conclude that T is a contraction since by equation (6), $\left(\sum_{j=0}^{k}\frac{\|c_j\|_\infty}{\Gamma(\beta-\delta_j+1)}+\frac{\Pi\Psi}{\Gamma(\beta+1)}\right) < 1$. By Banach contraction principle, T has a unique solution ξ in $C(Q,\mathbb{R})$.

Theorem 3.2. ($Convergence\ of\ Solution$). *If the solution is convergent, then it converges to the exact solution of the Volterra fractional integro-differential equation.*

Proof. Let S_u, S_v be arbitrary partial sums with $v \leq u$. We show that S_u is a Cauchy sequence. Let $S_u = \sum_{j=0}^{u} \xi_j(x)$ and $S_v = \sum_{j=0}^{v} \xi_j(x)$. Since $v \leq u$, then, we have from equation (5)

$$\begin{aligned}
S_u - S_v &= \sum_{j=v+1}^{u} \xi_j(x) \\
&= \sum_{j=0}^{k} \frac{1}{\Gamma(\beta-\delta_j)} \int_0^x (x-\sigma)^{\beta-\delta_j-1} c_j(\sigma) \sum_{j=v+1}^{u} \xi_j(\sigma)\, d\sigma \\
&\quad + \frac{1}{\Gamma(\beta)} \int_0^x (x-\sigma)^{\beta-1} \left(\int_0^\sigma K(\sigma,\tau) H\left(\sum_{j=v+1}^{u} \xi_j(\tau) \right) d\tau \right) d\sigma. \qquad (8)
\end{aligned}$$

Let $H\left(\sum_{j=v+1}^{u} \xi_j(\tau)\right) = \sum_{j=v+1}^{u} G_j(\tau)$, then equation (8) becomes

$$\begin{aligned}
|S_u - S_v| &= \left| \sum_{j=v+1}^{u} \xi_j(x) \right| \\
&= \left| \sum_{j=0}^{k} \frac{1}{\Gamma(\beta-\delta_j)} \int_0^x (x-\sigma)^{\beta-\delta_j-1} c_j(\sigma) \sum_{j=v+1}^{u} \xi_j(\sigma)\, d\sigma \right. \\
&\quad \left. + \frac{1}{\Gamma(\beta)} \int_0^x (x-\sigma)^{\beta-1} \left(\int_0^\sigma K(\sigma,\tau) \sum_{j=v+1}^{u} G_j(\tau)\, d\tau \right) d\sigma \right|. \qquad (9)
\end{aligned}$$

If, we let $\sum_{j=v}^{u-1} \xi_j(x) = S_{u-1} - S_{v-1}$, $\sum_{j=v}^{u-1} G_j(t) = H(S_{u-1}) - H(S_{v-1})$ in equation (9), then we have

$$\begin{aligned}
\|S_u - S_v\|_\infty &\leq \max_{\forall x \in Q} \left(\left| \sum_{j=0}^{k} \frac{1}{\Gamma(\beta-\delta_j)} \int_0^x (x-\sigma)^{\beta-\delta_j-1} c_j(\sigma) (S_{u-1} - S_{v-1})\, d\sigma \right. \right. \\
&\quad \left. \left. + \frac{1}{\Gamma(\beta)} \int_0^x (x-\sigma)^{\beta-1} \left(\int_0^\sigma K(\sigma,\tau) (H(S_{u-1}) - H(S_{v-1}))\, d\tau \right) d\sigma \right| \right) \\
&\leq \max_{\forall x \in Q} \left(\sum_{j=0}^{k} \frac{1}{\Gamma(\beta-\delta_j)} \int_0^x (x-\sigma)^{\beta-\delta_j-1} |c_j(\sigma)| |S_{u-1} - S_{v-1}|\, d\sigma \right. \\
&\quad \left. + \frac{1}{\Gamma(\beta)} \int_0^x (x-\sigma)^{\beta-1} \left(\int_0^\sigma |K(\sigma,\tau)| |H(S_{u-1}) - H(S_{v-1})|\, d\tau \right) d\sigma \right) \\
&\leq \sum_{j=0}^{k} \frac{\|c_j\|_\infty \|S_{u-1} - S_{v-1}\|_\infty}{\Gamma(\beta-\delta_j)} \int_0^x (x-\sigma)^{\beta-\delta_j-1}\, d\sigma \\
&\quad + \frac{\Pi\Psi \|S_{u-1} - S_{v-1}\|_\infty}{\Gamma(\beta)} \int_0^x (x-\sigma)^{\beta-1}\, d\sigma \\
&\leq \left(\sum_{j=0}^{k} \frac{\|c_j\|_\infty}{\Gamma(\beta-\delta_j+1)} + \frac{\Pi\Psi}{\Gamma(\beta+1)} \right) \|S_{u-1} - S_{v-1}\|_\infty.
\end{aligned}$$

Thus,

$$\|S_u - S_v\|_\infty \leq \rho \|S_{u-1} - S_{v-1}\|_\infty. \qquad (10)$$

where $\rho = \sum_{j=0}^{k} \frac{\|c_j\|_\infty}{\Gamma(\beta-\delta_j+1)} + \frac{\Pi\Psi}{\Gamma(\beta+1)}$.
Observe that, from equation (10)

$$\|S_u - S_v\|_\infty \leq \rho \|S_{u-1} - S_{v-1}\|_\infty \leq \rho \|S_{u-2} - S_{v-2}\|_\infty \leq \ldots \leq \rho \|S_1 - S_0\|_\infty$$

Also from equation (10)

$$\|S_{u-1}-S_{v-1}\|_\infty \leq \rho\,\|S_{u-2}-S_{v-2}\|_\infty\,,\ \|S_{u-2}-S_{v-2}\|_\infty \leq \rho\,\|S_{u-3}-S_{v-3}\|_\infty\,,\cdots$$

Therefore,

$$\|S_u-S_v\|_\infty \leq \rho\,\|S_{u-1}-S_{v-1}\|_\infty \leq \rho^2\,\|S_{u-2}-S_{v-2}\|_\infty \leq \ldots \leq \rho^v\,\|S_1-S_0\|_\infty \tag{11}$$

Let $u=v+1$, accordingly in equation (11) then we have

$$\|S_u-S_v\|_\infty \leq \rho\,\|S_v-S_{v-1}\|_\infty \leq \rho^2\,\|S_{v-1}-S_{v-2}\|_\infty \leq \ldots \leq \rho^v\,\|S_1-S_0\|_\infty\,.$$

That is,

$$\|S_p-S_q\|_\infty \leq \rho^v\,\|S_1-S_0\|_\infty \tag{12}$$

$\|S_u-S_v\|_\infty$ can be written as follows

$$\begin{aligned}
\|S_u-S_v\|_\infty &= \|S_{v+1}-S_v+S_{v+2}-S_{v+1}+S_{v+3}-S_{v+2}+S_{v+4}-S_{v+3}\\
&\quad +\cdots+S_{v+(u-v-2)}-S_{v+(u-v-1)}+S_{v+(u-v)}-S_{v+(u-v-1)}\|_\infty\\
&= \|S_{v+1}-S_v+S_{v+2}-S_{v+1}+S_{v+3}-S_{v+2}+S_{v+4}-S_{v+3}\\
&\quad +\cdots+S_{u-2}-S_{u-1}+S_u-S_{u-1}\|_\infty\\
&\leq \|S_{v+1}-S_v\|_\infty+\|S_{v+2}-S_{v+1}\|_\infty+\|S_{v+3}-S_{v+2}\|_\infty+\\
&\quad \|S_{v+4}-S_{v+3}\|_\infty+\cdots+\|S_{u-2}-S_{u-1}\|_\infty+\|S_u-S_{u-1}\|_\infty\,.
\end{aligned} \tag{13}$$

From equation (12) , let $u=v+1$, then

$$\begin{aligned}
\|S_{v+1}-S_v\|_\infty &\leq \rho^v\,\|S_1-S_0\|_\infty\\
\|S_{v+2}-S_{v+1}\|_\infty &\leq \rho^{v+1}\,\|S_1-S_0\|_\infty\\
\|S_{v+3}-S_{v+2}\|_\infty &\leq \rho^{v+2}\,\|S_1-S_0\|_\infty\\
&\ \ \vdots\\
\|S_u-S_{u-1}\|_\infty &\leq \rho^{u-1}\,\|S_1-S_0\|_\infty\,.
\end{aligned}$$

Therefore, equation (13) can be written as

$$\begin{aligned}
\|S_u-S_v\|_\infty &\leq \left(\rho^v+\rho^{v+1}+\rho^{v+2}+\cdots+\rho^{u-1}\right)\|S_1-S_0\|_\infty\\
&= \rho^v\left(1+\rho^1+\rho^2+\cdots+\rho^{u-v-1}\right)\|S_1-S_0\|_\infty\,.
\end{aligned}$$

By geometric series, let $q=u-v-1$, this implies,

$$\|S_u-S_v\|_\infty \leq \rho^v\left(\frac{1-\rho^{u-v}}{1-\rho}\right)\|S_1-S_0\|_\infty$$

since $0<\rho<1$, this means $1-\rho^{u-v}<1$, then

$$\|S_u-S_v\|_\infty \leq \frac{\rho^v}{1-\rho}\,\|\xi_1\|_\infty\,.$$

But $|\xi_1(x)|<\infty$ and $\lim\limits_{v\longrightarrow\infty}\frac{\rho^v}{1-\rho}=0$, since $\rho^v\longrightarrow 0$ as $v\longrightarrow\infty$. Therefore, $\|S_u-S_v\|_\infty\longrightarrow 0$ as $v\longrightarrow\infty$. We conclude that S_u is a Cauchy sequence in $C[0,1]$. Therefore, $\lim\limits_{n\longrightarrow\infty}\xi_n=\xi$. Thus, the solution is convergent.

4 Examples

Example 1 [39]. Consider the fractional order Volterra type integro-differential equation

$$D^{\frac{3}{4}}\xi(x)+\frac{x^2e^x}{5}\xi(x)-\int_0^x e^x\tau\xi(\tau)\,d\tau=\frac{6x^{\frac{9}{4}}}{\Gamma\left(\frac{13}{4}\right)} \tag{14}$$

Subject to $\xi(0)=0$ with exact solution $\xi(x)=x^3$.

Solution

Following Theorem 1, we have

$$\begin{aligned}
\left|(T\xi_2)(x)-(T\xi_1)(x)\right| &= \left|-\frac{1}{5\Gamma\left(\frac{3}{4}\right)}\int_0^x (x-\sigma)^{\frac{-1}{4}}\sigma^2e^\sigma\left(\xi_2(\sigma)-\xi_1(\sigma)\right)d\sigma\right. \\
&\quad \left.+\frac{1}{\Gamma\left(\frac{3}{4}\right)}\int_0^x (x-\sigma)^{\frac{-1}{4}}\left(\int_0^\sigma e^\sigma\tau\left(\xi_2(\tau)-\xi_1(\tau)\right)d\tau\right)d\sigma\right| \\
&\le \left|\frac{1}{5\Gamma\left(\frac{3}{4}\right)}\int_0^x (x-\sigma)^{\frac{-1}{4}}\sigma^2e^\sigma\left(\xi_2(\sigma)-\xi_1(\sigma)\right)d\sigma\right|+ \\
&\quad \left|\frac{1}{\Gamma\left(\frac{3}{4}\right)}\int_0^x (x-\sigma)^{\frac{-1}{4}}\left(\int_0^\sigma e^\sigma\tau\left(\xi_2(\tau)-\xi_1(\tau)\right)d\tau\right)d\sigma\right| \\
&\le \frac{1}{5\Gamma\left(\frac{3}{4}\right)}\int_0^x (x-\sigma)^{\frac{-1}{4}}\sigma^2e^\sigma\left|\xi_2(\sigma)-\xi_1(\sigma)\right|d\sigma+ \\
&\quad \frac{1}{\Gamma\left(\frac{3}{4}\right)}\int_0^x (x-\sigma)^{\frac{-1}{4}}\left(\int_0^\sigma e^\sigma\tau\left|\xi_2(\tau)-\xi_1(\tau)\right|d\tau\right)d\sigma \\
&\le \frac{\|\xi_2-\xi_1\|_\infty}{5\Gamma\left(\frac{3}{4}\right)(n)!}\int_0^x (x-\sigma)^{-\frac{1}{4}}\sigma^{n+2}d\sigma+ \\
&\quad \frac{\|\xi_2-\xi_1\|_\infty}{2\Gamma\left(\frac{3}{4}\right)(n)!}\int_0^x (x-\sigma)^{-\frac{1}{4}}\sigma^3d\sigma \\
&\le \left(\frac{\Gamma(4)\,x^{3.75}}{5\Gamma(4.75)}+\frac{\Gamma(4)\,x^{3.75}}{2\Gamma(4.75)}\right)\|\xi_2-\xi_1\|_\infty .
\end{aligned}$$

Thus,

$$\|T\xi_2-T\xi_1\|_\infty\le(0.253\,22)\,\|\xi_2-\xi_1\|_\infty .$$

Since $0.253\,22<1$, we say that the problem satisfies the condition of $Theorem$ 1.

Example 2 [40]. Consider the multi fractional order differential equation of the form

$$aD^2\xi(x)+b(x)D^{\beta_1}\xi(x)+c(x)D\xi(x)+e(x)D^{\beta_2}\xi(x)+k(x)\xi(x)+\lambda\int_0^x x\tau\xi(\tau)\,d\tau=f(x). \tag{15}$$

Subject to $\xi(0)=2,\ \xi'(0)=0$ with exact solution $\xi(x)=2-\frac{x^2}{2}$,

where $f(x)=-a-\frac{b(x)}{\Gamma(3-\gamma_1)}x^{2-\gamma_1}-c(x)x-\frac{e(x)}{\Gamma(3-\gamma_2)}x^{2-\gamma_2}+k(x)\left(2-\frac{x^2}{2}\right)$, $a=1,\ b(x)=x^{\frac{1}{2}},\ c(x)=x^{\frac{1}{3}},\ e(x)=x^{\frac{1}{4}},\ k(x)=x^{\frac{1}{5}},\ \lambda=0,\ \gamma_2=0.333$ and $\gamma_1=1.234$.

Solution

Following Theorem 1, we have

$$
\begin{aligned}
\left|(T\xi_2)(x)-(T\xi_1)(x)\right| &= \Bigg|-\frac{1}{\Gamma(2)\Gamma(1.766)}\int_0^x (x-\sigma)\,\sigma^{1.266}\left(\xi_2(\sigma)-\xi_1(\sigma)\right)d\sigma \\
&\quad -\frac{1}{\Gamma(2)}\int_0^x (x-\sigma)\,\sigma^{1.333}\left(\xi_2(\sigma)-\xi_1(\sigma)\right)d\sigma \\
&\quad -\frac{1}{\Gamma(2)\Gamma(1.667)}\int_0^x (x-\sigma)\,\sigma^{0.917}\left(\xi_2(\sigma)-\xi_1(\sigma)\right)d\sigma \\
&\quad -\frac{1}{\Gamma(2)}\int_0^x (x-\sigma)\,\sigma^{0.2}\left(\xi_2(\sigma)-\xi_1(\sigma)\right)d\sigma\Bigg| \\
&\le \frac{1}{\Gamma(2)\Gamma(1.766)}\int_0^x (x-\sigma)\,\sigma^{1.266}\left|\xi_2(\sigma)-\xi_1(\sigma)\right|d\sigma \\
&\quad +\frac{1}{\Gamma(2)}\int_0^x (x-\sigma)\,\sigma^{1.333}\left|\xi_2(\sigma)-\xi_1(\sigma)\right|d\sigma \\
&\quad +\frac{1}{\Gamma(2)\Gamma(1.667)}\int_0^x (x-\sigma)\,\sigma^{0.917}\left|\xi_2(\sigma)-\xi_1(\sigma)\right|d\sigma \\
&\quad +\frac{1}{\Gamma(2)}\int_0^x (x-\sigma)\,\sigma^{0.2}\left|\xi_2(\sigma)-\xi_1(\sigma)\right|d\sigma \\
&\le \frac{\|\xi_2-\xi_1\|_\infty}{\Gamma(2)\Gamma(1.766)}\int_0^x (x-\sigma)\,\sigma^{1.266}d\sigma \\
&\quad +\frac{\|\xi_2-\xi_1\|_\infty}{\Gamma(2)}\int_0^x (x-\sigma)\,\sigma^{1.333}d\sigma \\
&\quad +\frac{\|\xi_2-\xi_1\|_\infty}{\Gamma(2)\Gamma(1.667)}\int_0^x (x-\sigma)\,\sigma^{0.917}d\sigma \\
&\quad +\frac{\|\xi_2-\xi_1\|_\infty}{\Gamma(2)}\int_0^x (x-\sigma)\,\sigma^{0.2}d\sigma
\end{aligned}
$$

By property (iii) we have

$$
\|T\xi_2-T\xi_1\|_\infty \le \left(\frac{\Gamma(2.266)}{\Gamma(1.766)\Gamma(4.266)}+\frac{\Gamma\left(\frac{7}{3}\right)}{\Gamma\left(\frac{13}{3}\right)}+\frac{\Gamma(1.917)}{\Gamma(1.667)\Gamma(3.917)}+\frac{\Gamma\left(\frac{6}{5}\right)}{\Gamma\left(\frac{16}{5}\right)}\right)\|\xi_2-\xi_1\|_\infty .
$$

Thus,

$$
\|T\xi_2-T\xi_1\|_\infty \le (0.851\,87)\,\|\xi_2-\xi_1\|_\infty .
$$

Since $0.851\,87<1$, we say that the problem satisfies the condition of the $Theorem$ 1.

5 Discussions and Conclusion

In this paper, a new class of multi-term fractional order Volterra integro-differential equation has been transformed to its equivalent integral form with the help of Riemann-Liouville integral of fractional order. Banach contraction principle was utilised in proving the uniqueness of the solution of the multi-term fractional order Volterra integro-differential equation. Furthermore, the convergence analysis was also studied and proved using the Chauchy convergence creteria. To demonstrate the applicability and solvability of the problem, examples were presented to prove that the condition is met. The results obtained were in perfect agreement with the condition of the uniqueness of solution theorem.

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