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Analysis of investment portfolio with 2×2 positive definite matrix and mortality risk under exponential utility

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Abstract

This paper investigates a pension plan member's (PPM) portfolio in a defined contributory (DC) pension plan with return clause and charge on balance exhibiting constant absolute risk averse (CARA). A portfolio comprising of a risk-free asset and two risky assets modelled by the geometric Brownian motion (GBM) process is considered, where the instantaneous volatilities of the two risky assets form a 2×2 matrix $k = \{k_{a,b}\}_{2\times2}$ such that kk^T is positive definite. The Abraham De Moivre mortality force function is use to determine the mortality rate of PPMs during accumulation phase and an optimization problem is obtained from the Hamilton Jacobi Bellman (HJB) equation using dynamic programming approach. Using Legendre transformation and dual theory with variable change technique, the optimal value function (OVF), optimal control plan (OCP) and the optimal fund size (OFS) are obtained under exponential utility function. Furthermore, some numerical analysis of some sensitive parameters such as risk free interest rate, risk averse coefficient, entry age of PPM, charge on balance and OFS are presented to explain their impact on the OCP.

Keywords and phrases: Abraham De Moivre mortality force function; optimal control plan; Legendre transform and dual theory; GBM process; exponential utility.

1 Introduction

In the defined contributory pension scheme, PPMs are mandated to contribute a certain proportion of their income into their retirement savings account (RSA) for the purpose of investment and to plan for the retirement needs of PPMs by the pension fund administrators (PFA) [6]. In order for the PFAs to maximize the PPM's wealth at the time of retirement, there is need to understand how investments are done for maximum returns with minimal risk; this has led to the study of OCP. A good number of works have been done in this regards under different assumption [1, 5, 7, 8, 9, 10, 12, 17, 22, 23, 26, 28, 29].

One of the major challenges faced by most PFAs in the DC pension scheme is the determination of investment strategy under mortality risk [11]. This is so, because they will be faced with the challenges of paying off the contributions due to PPMs who die during the accumulation period and at the same time managed the portfolio of the remaining PPMs. In achieving this, two mortality force functions have been used by some researchers to model the mortality rate of PPM in the DC pension scheme during the accumulation phase which include Weibull mortality force function and Abraham De Moivre force function [4, 11, 16, 19, 21, 24, 25]. However, a number of research works have been done in this regard. They include; [11], who studied OCP with return of premium clauses under mean variance utility. In their research work, they consider a PPM with a portfolio consisting of one risk free asset and a risky asset modelled by GBM and used the game theoretic approach to determine the optimal investment strategy and the efficient frontier of the PPM. [24], investigated studied the optimal investment problem with return clause for a PPM having a portfolio comprising of one risk free asset and a risky asset modelled by Heston volatility model; they also used the mean variance utility and

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the same method in [11] to determine the OCP and the efficient frontier. In [19], the default risk was introduce into the model in [11], where pension funds are allowed to be invested in a risk-free asset, a default able bond and a risky asset whose price process follows the CEV model. They used the variational method to obtain the OCP and the efficient frontier. [4] studied the OCP with return clause; in their work, they considered investment in three different assets comprising of fixed deposit, stock and loan and used the game theoretic approach to determine the OCP and the efficient frontier under mean variance utility. [25], studied the same problem in [11] but used the Jump diffusion process to model the risky asset and determine the OCP and efficient frontier. Also, [2, 3] investigated the OCP with return of premium with predetermined interest; in their research work, the returned premium is with interest from the invested fund in risk free asset. They went further to determine the OCP and the efficient frontier of the PPM. In all the research work mention above, the utility function used was the mean variance utility developed by Markowitz [20] and the mortality rate of the death PPM were determined using Abraham De Moivre force function.

Very recently, [16] studied the OCP with return clause under power utility function; in their work, they considered a PPM with a portfolio consisting of one risk free asset and a risky asset which follows the constant elasticity of variance model. Also, they used the Weibull mortality force function to model the mortality rate of members who died during the accumulation phase and determine the OCP of the PPM. In [21], the OCP with return clause was studied under Weibull mortality force function. They took into consideration tax on the invested funds.

The main contribution of this paper is that we choose a portfolio consisting of three different assets, one risk free asset and two risky asset which forms a 2×2 matrix which is positive definite similar to one in [18]. Also, the Abraham De Moivre force function is used to determine the mortality rate of the PPM during the accumulation phase and the charge on balance on the PPM portfolio is also considered. The Legendre transformation method and dual theory is applied to solve for the OCP, the OVF and OFS which are not captured in the literature for a PPM with return of premium exhibiting constant absolute risk averse. Finally, we study the impact of some sensitive parameters on OCP. More precisely, the impact of PPM's entry age on the optimal control plan and optimal fund size for investment are studied and the impact of the charge on balance on the OCP.

2 The investment model

In this section, we shall consider a financial market which is continuously open for an interval $t \in [0,\ T]$ where $T > 0$, is the expiration date of the investment. Also, we shall consider a portfolio with of one risk free asset and two risky assets. Let $\{\mathscr{W}_1(t),\ \mathscr{W}_2(t),\ t\ =\ 0\}$ be two standard Brownian motion defined on a complete probability space $(\Omega, \mathscr{F}, \mathscr{P})$ where Ω is a real space and $\mathscr{P}$ is a probability measure and $\mathscr{F}$ is the filtration which represents the information generated by the two Brownian motions.

Let $\mathscr{N}_t^0\,(t)$ denote the price of the risk free asset at time t and the model is given as follows

$$d\mathscr{N}_t^0\,(t) = r\mathscr{N}_t^0\,(t)\,dt\mathscr{N}_t^0\,(t) = n_0 > 0 \tag{2.1}$$

where $r(t)$ is the predetermine interest such that $r > 0$

Let $\mathscr{N}_t^1\,(t)$ and $\mathscr{N}_t^2\,(t)$ be the prices of stock 1 and stock 2 respectively which are modelled by geometric Brownian motions model and the dynamics of the stock market prices are given by stochastic differential equations at $t \geq 0$ as follows

$$\begin{pmatrix} \frac{d\mathscr{N}_t^1(t)}{\mathscr{N}_t^1(t)} \\ \frac{d\mathscr{N}_t^2(t)}{\mathscr{N}_t^2(t)} \end{pmatrix} = \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} dt + \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} d\mathscr{W}_1 \\ d\mathscr{W}_2 \end{pmatrix} \tag{2.2}$$

where ℓ_1 and ℓ_2 represent the expectated appreciation rate of stock 1 and 2 respectively. $m_{11},\ m_{12},\ m_{21},\ m_{22}$ represent the instantaneous volatilities and form a 2×2 matrix $m = \{m_{ab}\}_{2\times 2}$ such that mm^T give a positive definite matrix [18].

3 Optimization problem

3.1 Wealth formulations and HJB equation

Let $k = (k_0,\ k_1,\ k_2)$ represents be the optimal control plan (OCP), k_1 is the proportion of investment in risk free asset and $k_1$and k_2 are proportion of investment in the two risky assets. Next, we define the utility attained by the investor from a given state h at time t as

$$\mathscr{F}_k\left(t,h\right) = E_k\ U\left(\mathscr{H}\left(T\right)\right) \mid \mathscr{H}\left(t\right) = h\right], \tag{3.1}$$

where t is the time and h is the wealth of the pension plan member. Our interest here is to obtain the OCP and the optimal value function of the PPM given by

$$k^* \text{ and } \mathscr{F}\left(\mathrm{t,h}\right) = \sup_{\mathrm{k}} \mathscr{F}_{\mathrm{k}}\left(\mathrm{t,h}\right) \tag{3.2}$$

Respectively such that

$$\mathscr{F}_{k^*}\left(t,h\right) = \mathscr{F}\left(t,h\right). \tag{3.3}$$

Since our interest is to maximize $\mathscr{F}_k\left(t,h\right)$ subject to the PPM's wealth denoted by $\mathscr{H}\left(t\right)$, we proceed to formulate our wealth function as follows

Let c be the mandatory monthly contribution paid by the PPM at time t, e_0 the initial age of accumulation phase, T the accumulation phase period such that $e_0 + T$ is the terminal age of the PPM. $\aleph_{\frac{1}{\imath},e_0+t}$ is the mortality rate from time t to $t+\frac{1}{\imath}$, tc is the accumulated contributions at time t, $tc\aleph_{\frac{1}{\imath},e_0+t}$ is the returned contributions to the next of kin during the accumulation period.

Corresponding to the OCP $k = (k_0,\ k_1,\ k_2)$, where $k_0 = 1 - k_1 - k_2$ and the accumulation phase period $[t, t+\frac{1}{\imath}]$, the stochastic differential form for the wealth function similar to [11] is given as:

$$\mathscr{H}\left(t+\frac{1}{\imath}\right) = \left[\mathscr{H}\left(t\right)\left(k_0\frac{\mathscr{N}^0_{t+\frac{1}{\imath}}}{\mathscr{N}^0_t} + k_1\frac{\mathscr{N}^1_{t+\frac{1}{\imath}}}{\mathscr{N}^1_t} + k_2\frac{\mathscr{N}^2_{t+\frac{1}{\imath}}}{\mathscr{N}^2_t}\right) + c\frac{1}{\imath} - tc\aleph_{\frac{1}{\imath},e_0+t}\right]\left(\frac{1}{1-\aleph_{\frac{1}{\imath},e_0+t}}\right) \tag{3.4}$$

$$\mathscr{H}\left(t+\frac{1}{\imath}\right) - \mathscr{H}\left(t\right) = \left[\begin{array}{c}\mathscr{H}\left(t\right)\left(\begin{array}{c}(1-k_1-k_2)\left(\frac{\mathscr{N}^0_{t+\frac{1}{\imath}}-\mathscr{N}^0_t}{\mathscr{N}^0_t}\right)\\ +k_1\left(\frac{\mathscr{N}^1_{t+\frac{1}{\imath}}-\mathscr{N}^1_t}{\mathscr{N}^1_t}\right) + k_2\left(\frac{\mathscr{N}^2_{t+\frac{1}{\imath}}-\mathscr{N}^2_t}{\mathscr{N}^2_t}\right)\end{array}\right)\\ +c\frac{1}{\imath} - tc\aleph_{\frac{1}{\imath},e_0+t}\end{array}\right]\left(1+\frac{\aleph_{\frac{1}{\imath},e_0+t}}{1-\aleph_{\frac{1}{\imath},e_0+t}}\right) \tag{3.5}$$

From [11],

$$\begin{cases} c\frac{1}{\imath} \to cdt,\ \frac{\mathscr{N}^0_{t+\frac{1}{\imath}}-\mathscr{N}^0_t}{\mathscr{N}^0_t} \to \frac{d\mathscr{N}^0_t}{\mathscr{N}^0_t},\ \frac{\mathscr{N}^1_{t+\frac{1}{\imath}}-\mathscr{N}^1_t}{\mathscr{N}^1_t} \to \frac{d\mathscr{N}^1_t}{\mathscr{N}^1_t},\ \frac{\mathscr{N}^2_{t+\frac{1}{\imath}}-\mathscr{N}^2_t}{\mathscr{N}^2_t} \to \frac{d\mathscr{N}^2_t}{\mathscr{N}^2_t},\\ \frac{1}{\imath} \to 0\ as\ \imath \to \infty,\ \aleph_{\frac{1}{\imath},e_0+t} = \emptyset\left(e_0+t\right)dt,\ \frac{\aleph_{\frac{1}{\imath},e_0+t}}{1-\aleph_{\frac{1}{\imath},e_0+t}} = \emptyset\left(e_0+t\right)dt, \end{cases} \tag{3.6}$$

Substituting (3.6) into (3.5), we have

$$d\mathscr{H}\left(t\right) = \left[\begin{array}{c}\mathscr{H}\left(t\right)\left(k_1\frac{d\mathscr{N}^1_t}{\mathscr{N}^1_t} + k_2\frac{d\mathscr{N}^2_t}{\mathscr{N}^2_t} + (1-k_1-k_2)\frac{d\mathscr{N}^0_t}{\mathscr{N}^0_t}\right)\\ +cdt - tc\emptyset\left(e_0+t\right)dt\end{array}\right]\left(1+\emptyset\left(e_0+t\right)dt\right) \tag{3.7}$$

Substituting (2.1) and (2.2) into (3.7), we have

$$d\mathscr{H}\left(t\right) = \left[\begin{array}{c}\mathscr{H}\left(t\right)\left(\begin{array}{c}\left[k_1\left(\ell_1-r\right) + k_2\left(\ell_2-r\right) + \left(r-\vartheta\right) + \emptyset\left(e_0+t\right)\right]dt\\ +(k_1m_{11}+k_2m_{21})d\mathscr{W}_1 + (k_1m_{12}+k_2m_{22})d\mathscr{W}_2\end{array}\right)\\ +c\left(1-t\emptyset\left(e_0+t\right)\right)dt\end{array}\right] \tag{3.8}$$

Since $\emptyset(t)$ is the force function and e is the maximal age of the life table. From [11] [15], the Abraham De Moivre mortality force function formula is given as

$$\emptyset(t) = \frac{1}{(e-t)} \quad 0 \leq t < e. \tag{3.9}$$

This implies that

$$\emptyset(e_0 + t) = \frac{1}{(e - e_0 - t)} \tag{3.10}$$

Substituting (3.10) into (3.8) and simplifying it, we have

$$d\mathscr{H}(t) = \left[\begin{array}{c} \mathscr{H}(t) \left(\begin{array}{c} \left[k_1(\ell_1 - r) + k_2(\ell_2 - r) + \left(r - \vartheta + \frac{1}{e-e_0-t} \right) \right] dt \\ +(k_1 m_{11} + k_2 m_{21}) d\mathscr{W}_1 + (k_1 m_{12} + k_2 m_{22}) d\mathscr{W}_2 \end{array} \right) \\ +c\left(\frac{e-e_0-2t}{e-e_0-t} \right) dt \\ \mathscr{H}(0) = h_0 \end{array} \right] \tag{3.11}$$

From [13], we apply the Ito's lemma and maximum principle theorem to derive the Hamilton Jacobi Bellman (HJB) equation which is a nonlinear PDE associated with (3.11). This is obtained by maximizing (3.1) subject to (3.11) as follows

$$\left\{ \begin{array}{c} \mathscr{F}_t + \left[\left(r - \vartheta + \frac{1}{e-e_0-t} \right) h + c\left(\frac{e-e_0-2t}{e-e_0-t} \right) \right] \mathscr{F}_h \\ +\sup_{k_1,\, k_2} \left[\begin{array}{c} \left(\begin{array}{c} \frac{1}{2}\left(m_{11}^2 + m_{12}^2 \right) k_1^2 + (m_{11} m_{21} + m_{12} m_{22}) k_1 k_2 \\ +\frac{1}{2}\left(m_{21}^2 + m_{22}^2 \right) k_2^2 \end{array} \right) \mathscr{F}_{hh} \\ + (k_1(\ell_1 - r) + k_2(\ell_2 - r)) \mathscr{F}_h \end{array} \right] \end{array} \right\} = 0 \tag{3.12}$$

If $m_{11}^2 + m_{12}^2 = \mathscr{P}$, $m_{21}^2 + m_{22}^2 = \mathscr{Q}$ and $m_{11}m_{21} + m_{12}m_{22} = \mathscr{R}$, then we (3.12) becomes

$$\left\{ \begin{array}{c} \mathscr{F}_t + \left[\left(r - \vartheta + \frac{1}{e-e_0-t} \right) h + c\left(\frac{e-e_0-2t}{e-e_0-t} \right) \right] \mathscr{F}_h \\ +\sup_{k_1,\, k_2} \left[\begin{array}{c} \left(\frac{1}{2} k_1^2 \mathscr{P} + k_1 k_2 \mathscr{R} + \frac{1}{2} k_2^2 \mathscr{Q} \right) \mathscr{F}_{hh} \\ + (k_1(\ell_1 - r) + k_2(\ell_2 - r)) \mathscr{F}_h \end{array} \right] \end{array} \right\} = 0 \tag{3.13}$$

Next, we differentiate (3.13) with respect to k_1 *and* k_2, to obtain the first order maximizing condition for equation (3.13) as

$$k_1^* = -\frac{\mathscr{R}(\ell_2 - r) - \mathscr{Q}(\ell_1 - r)}{h(\mathscr{P}\mathscr{Q} - \mathscr{R}^2)} \frac{\mathscr{F}_h}{\mathscr{F}_{hh}} \tag{3.14}$$

$$k_2^* = -\frac{\mathscr{R}(\ell_1 - r) - \mathscr{P}(\ell_2 - r)}{h(\mathscr{P}\mathscr{Q} - \mathscr{R}^2)} \frac{\mathscr{F}_h}{\mathscr{F}_{hh}} \tag{3.15}$$

Substituting (3.14) and (3.15) into (3.13), we have

$$\left\{ \mathscr{F}_t + \frac{1}{2}(\mathscr{C} - \mathscr{D} - \mathscr{E}) \frac{\mathscr{F}_h^2}{\mathscr{F}_{hh}} + \left[\left(r - \vartheta + \frac{1}{e - e_0 - t} \right) h + c\left(\frac{e - e_0 - 2t}{e - e_0 - t} \right) \right] \mathscr{F}_h \right\} = 0 \tag{3.16}$$

where

$$\left\{ \mathscr{C} = \frac{2\mathscr{R}(\mu_1 - r)(\mu_2 - r)}{(\mathscr{P}\mathscr{Q} - \mathscr{R}^2)}, \mathscr{D} = \frac{\mathscr{Q}(\mu_1 - r)^2}{(\mathscr{P}\mathscr{Q} - \mathscr{R}^2)}, \mathscr{E} = \frac{\mathscr{P}(\mu_2 - r)^2}{(\mathscr{P}\mathscr{Q} - \mathscr{R}^2)} \right. \tag{3.17}$$

3.2 Legendre transformation and dual theory

We can observed that (3.16) is a non linear PDE and is some worth difficult to solve. This section, intend to introduce the Legendre transformation and dual theory and apply it to transform the non linear PDE to a linear PDE.

Theorem 3.1. *Let $g : R^n \to R$ be a convex function for $z > 0$, define the Legendre transform*

$$\mathscr{M}(x) = \max_h \{g(h) - zh\}, \tag{3.18}$$

where $\mathscr{M}(x)$ is the Legendre dual of $y(h)$.

Since $y(h)$ is convex, then from theorem 3.1 and [14, 27], the Legendre transform for the value function $\mathscr{F}(t, h)$ is defined thus

$$\widehat{\mathscr{F}}(t, x) = \sup \mathscr{F}(t, h) - xh \mid 0 < h < \infty\} \quad 0 < t < T \tag{3.19}$$

where $\widehat{\mathscr{F}}$ is the dual of $\mathscr{F}$ and $x > 0$ is the dual variable of x.
The value of x where this optimum is achieved is represented by $y(t, z)$, such that

$$y(t, x) = \inf h \,\Big|\, \mathscr{F}(t, h) \geq xy + \widehat{\mathscr{F}}\Big\} \quad 0 < t < T. \tag{3.20}$$

From (3.20), the function y and $\widehat{\mathscr{F}}$ are very much related and can be refers to as the dual of $\mathscr{F}$ and are related thus

$$\widehat{\mathscr{F}}(t, x) = \mathscr{F}(t, y) - xy. \tag{3.21}$$

where

$$y(t, x) = h, \; \mathscr{F}_h = x, \; y = -\widehat{\mathscr{F}}_x. \tag{3.22}$$

Differentiating (3.22) with respect to t andh, we have

$$\begin{cases} \mathscr{F}_t = \widehat{\mathscr{F}}_t, \; \mathscr{F}_h = x, \\ \mathscr{F}_{hh} = \frac{-1}{\widehat{\mathscr{F}}_{xx}}, \\ , \end{cases} \tag{3.23}$$

Substituting (3.23) into (3.14), (3.15) and (3.16), we have

$$k_1^* = -\frac{\mathscr{R}(\ell_2 - r) - \mathscr{Q}(\ell_1 - r)}{y(\mathscr{P}\mathscr{Q} - \mathscr{R}^2)} x\widehat{\mathscr{F}}_{xx} \tag{3.24}$$

$$k_2^* = -\frac{\mathscr{R}(\ell_1 - r) - \mathscr{P}(\ell_2 - r)}{y(\mathscr{P}\mathscr{Q} - \mathscr{R}^2)} x\widehat{\mathscr{F}}_{xx} \tag{3.25}$$

$$\left\{\mathscr{F}_t - \frac{1}{2}(\mathscr{C} - \mathscr{D} - \mathscr{E}) x^2 \widehat{\mathscr{F}}_{xx} + \left[\left(r - \vartheta + \frac{1}{e - e_0 - t}\right) y + c\left(\frac{e - e_0 - 2t}{e - e_0 - t}\right)\right] x\right\} = 0 \tag{3.26}$$

If we differentiate (3.24), (3.25) and (3.26) with respect to x, we will have

$$\left\{ \begin{array}{c} y_t + \frac{1}{2}(\mathscr{C} - \mathscr{D} - \mathscr{E}) x^2 y_{xx} - \left[\left(r - \vartheta + \frac{1}{e - e_0 - t}\right) y + c\left(\frac{e - e_0 - 2t}{e - e_0 - t}\right)\right] \\ -\left(r - \vartheta + \frac{1}{e - e_0 - t} - \frac{1}{2}(\mathscr{C} - \mathscr{D} - \mathscr{E})\right) x y_x \end{array} \right\} = 0 \tag{3.27}$$

$$k_1^* = -\frac{\mathscr{R}(\ell_2 - r) - \mathscr{Q}(\ell_1 - r)}{y(\mathscr{P}\mathscr{Q} - \mathscr{R}^2)} x y_x \tag{3.28}$$

$$k_2^* = -\frac{\mathscr{R}(\ell_1 - r) - \mathscr{P}(\ell_2 - r)}{y(\mathscr{P}\mathscr{Q} - \mathscr{R}^2)} x y_x \tag{3.29}$$

Next, we proceed to solve (3.27) for y considering PPM with exponential utility using change of variable, after which we will use the solution of y to simplify the OCPs in (3.28) and (3.29).

4 OCP with return clause and charge on balance under exponential utility

In this section, we define our utility function in terms of the dual function. Let $U(h)$ be the original utility function given as

$$U(h) = \frac{1}{\gamma}e^{-\gamma h} \tag{4.1}$$

And the value function is given in terms of the utility as

$$\mathscr{F}(t,h) = U(h)$$

At terminal time T, we define the dual utility in terms of the original utility function $U(h)$ as

$$\hat{U}(x) = \sup U(h) - hx \mid 0 < x < \infty\},$$

and

$$G(x) = \sup\{h|U(h) \geq xh + \hat{U}(x).$$

As a result $\widehat{\mathscr{F}}(t,x) = \mathscr{F}(t,y) - xy$.

$$G(x) = (U')^{-1}(x), \tag{4.2}$$

where G is the inverse of the marginal utility U
At terminal time T, we can define

$$y(T,x) = \inf_{x>0} h \,\Big|\, U(h) \geq xh + \widehat{\mathscr{F}}(t,x)\Big\} \; and \; \widehat{\mathscr{F}}(t,x) = \sup_{x>0}\{\, U(x) - xh\}$$

so that

$$y(T,x) = (U')^{-1}(x). \tag{4.3}$$

Proposition 4.1. *The optimal value function, optimal fund size and the optimal control plan are given as*

$$y(t,x) = \left(\begin{array}{c} -\frac{1}{\gamma}\left[\left(\frac{e-e_0-T}{e-e_0-t}\right) Exp[r-\vartheta](t-T) \left(\begin{array}{c} lnx \\ +\left(r-\vartheta-\frac{1}{2}(\mathscr{C}-\mathscr{D}-\mathscr{E})\right)(t-T) \\ +ln\left(\frac{e-e_0-T}{e-e_0-t}\right) \end{array} \right)\right] \\ +\frac{2(t-T)}{(r-\vartheta)(e-e_0-t)} Exp[r-\vartheta]t + \frac{2}{(r-\vartheta)^2(e-e_0-t)}\left[Exp[r-\vartheta](t-T)-1\right] \end{array} \right) \tag{4.4}$$

$$\mathscr{H}(t) = \left(\begin{array}{c} \frac{\left(2\mathscr{R}(\ell_1-r)(\ell_2-r)-\mathscr{P}(\ell_2-r)^2-\mathscr{Q}(\ell_1-r)^2\right)}{\gamma(\mathscr{P}\mathscr{Q}-\mathscr{R}^2)} \left[\begin{array}{c} \frac{tExp[r-\vartheta](t-T)}{e-e_0-t} \\ -\frac{T}{e-e_0-T} \end{array} \right] \\ +c\left(\frac{t-T}{(e-e_0-t)(e-e_0-T)}\right)\left(\frac{2r-2\vartheta+e-e_0}{(r-\vartheta)^2}\right) + h_0 \end{array} \right) \tag{4.5}$$

$$k_1^* = \frac{\mathscr{R}(\ell_2-r) - \mathscr{Q}(\ell_1-r)}{\gamma h_0(\mathscr{P}\mathscr{Q}-\mathscr{R}^2)}\left(\frac{e-e_0-T}{e-e_0-t}\right) Exp[r-\vartheta](t-T) \tag{4.6}$$

$$k_2^* = \frac{\mathscr{R}(\ell_1-r) - \mathscr{P}(\ell_2-r)}{\gamma h_0(\mathscr{P}\mathscr{Q}-\mathscr{R}^2)}\left(\frac{e-e_0-T}{e-e_0-t}\right) Exp[r-\vartheta](t-T) \tag{4.7}$$

Proof. From (4.1),

$$y(T,x) = (U')^{-1}(x) = -\frac{1}{\gamma}lnx \tag{4.8}$$

Next, we conjecture a solution to (3.27) similar to the one in [5, 30] with the form:

$$\begin{cases} y(t,x) = -\frac{1}{\gamma}[u(t)(lnx + v(t))] + w(t) \\ u(T) = 1, \; v(T) = 0, \quad w(T) = 0, \end{cases} \tag{4.9}$$

$$\left.\begin{array}{c} y_t = -\frac{1}{\gamma}[u_t\left(lnx + v(t)\right) + u\left(t\right)v_t] + w_t, \\ y_x = -\frac{1}{x\gamma}u\left(t\right), \\ y_{xx} = \frac{1}{x^2\gamma}u\left(t\right) \end{array}\right\} \tag{4.10}$$

Substituting (4.10) into (3.27), we have

$$\left\{\begin{array}{c} \left[u_t - \left(r - \vartheta + \frac{1}{e-e_0-t}\right)u\left(t\right)\right]lnx \\ -\left[w_t - \left(r - \vartheta + \frac{1}{e-e_0-t}\right)w\left(t\right) - c\left(\frac{e-e_0-2t}{e-e_0-t}\right)\right]\gamma \quad = 0 \\ -\frac{u(t)}{\gamma}\left[v_t - \left(r - \vartheta + \frac{1}{e-e_0-t}\right) + \frac{1}{2}\left(\mathscr{C} - \mathscr{D} - \mathscr{E}\right)\right] \end{array}\right. \tag{4.11}$$

Splitting (4.11) we have

$$\left\{\begin{array}{c} u_t - \left(r - \vartheta + \frac{1}{e-e_0-t}\right)u\left(t\right) = 0 \\ u(T) = 1 \end{array}\right. \tag{4.12}$$

$$\left\{\begin{array}{c} w_t - \left(r - \vartheta + \frac{1}{e-e_0-t}\right)w\left(t\right) - c\left(\frac{e-e_0-2t}{e-e_0-t}\right) = 0 \\ w\left(T\right) = 0 \end{array}\right. \tag{4.13}$$

$$\left\{\begin{array}{c} v_t - \left(r - \vartheta + \frac{1}{e-e_0-t}\right) + \frac{1}{2}\left(\mathscr{C} - \mathscr{D} - \mathscr{E}\right) = 0 \\ v\left(T\right) = 0 \end{array}\right. \tag{4.14}$$

Solving equation (4.12), (4.13) and (4.14) we have,

$$u(t) = \left(\frac{e - e_0 - T}{e - e_0 - t}\right)Exp\left[r - \vartheta\right]\left(t - T\right) \tag{4.15}$$

$$w\left(t\right) = \frac{2\left(t - T\right)}{\left(r - \vartheta\right)\left(e - e_0 - t\right)}Exp\left[r - \vartheta\right]t + \frac{2}{\left(r - \vartheta\right)^2\left(e - e_0 - t\right)}\left[Exp\left[r - \vartheta\right]\left(t - T\right) - 1\right] \tag{4.16}$$

$$v\left(t\right) = \left(r - \vartheta - \frac{1}{2}\left(\mathscr{C} - \mathscr{D} - \mathscr{E}\right)\right)\left(t - T\right) + ln\left(\frac{e - e_0 - T}{e - e_0 - t}\right) \tag{4.17}$$

Substituting (4.11), (4.12) and (4.13) into (4.9), we obtain (4.4)
Also, from (4.10), we have

$$y_x = -\frac{1}{x\gamma}\left(\frac{e - e_0 - T}{e - e_0 - t}\right)Exp\left[r - \vartheta\right]\left(t - T\right) \tag{4.18}$$

Substituting (4.18) in (3.28) and (3.29), we obtain (4.6) and (4.7)
Substituting (4.6) and (4.7) into (3.11) and solve the resultant differential equation, we obtain (4.5).

5 Numerical analysis

In this section, some numerical simulations are presented to analyze the impact of some sensitive parameters on the optimal control plans under exponential utility. To achieve this, the following data will be used unless otherwise stated; $r\left(0\right) = 0.07$, $T = 20$, $h_0 = 1$, $\vartheta = 0.02$, $m_{11} = 1$, $m_{12} = 0.85$, $m_{21} = 0.95$, $m_{22} = 0.8$, $, \ell_1 = 0.12$, $\ell_1 = 1.0$, $\gamma = 0.05$, $e = 100$, $e_0 = 20$.

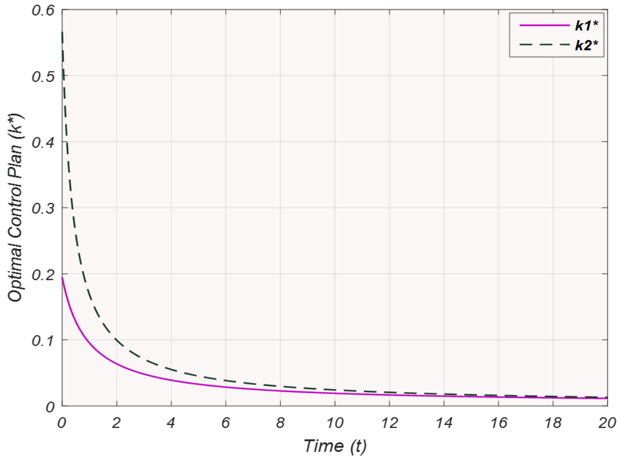

Figure 1: Optimal control plan for the two risky assets

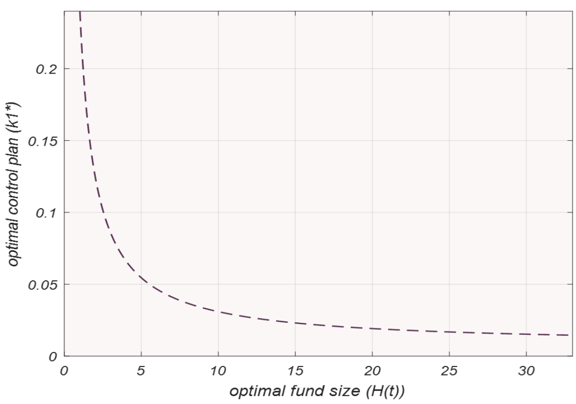

Figure 2: Impact of optimal fund size on optimal control plan k_1^*

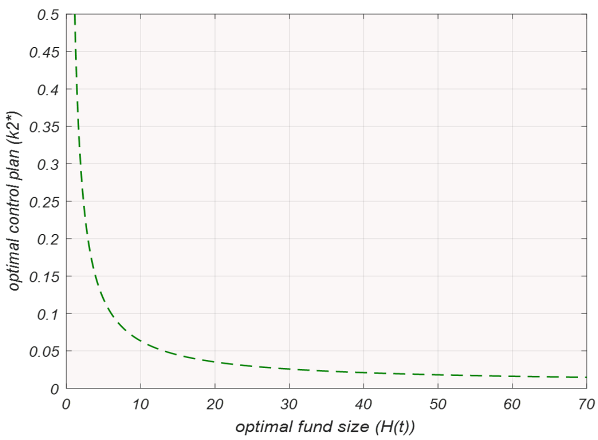

Figure 3: Impact of optimal fund size on optimal control plan k_2^*

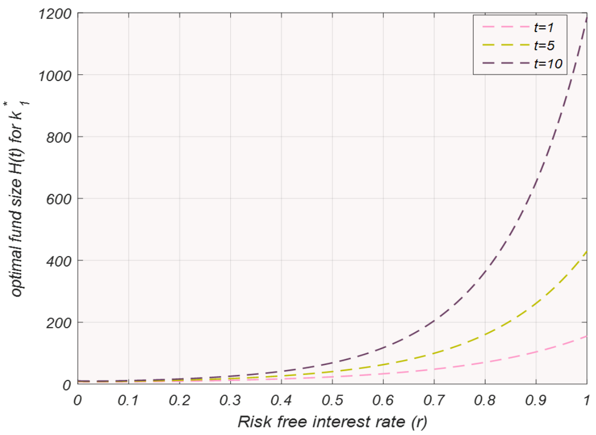

Figure 4: Impact of risk free interest on optimal control plan k_1^*

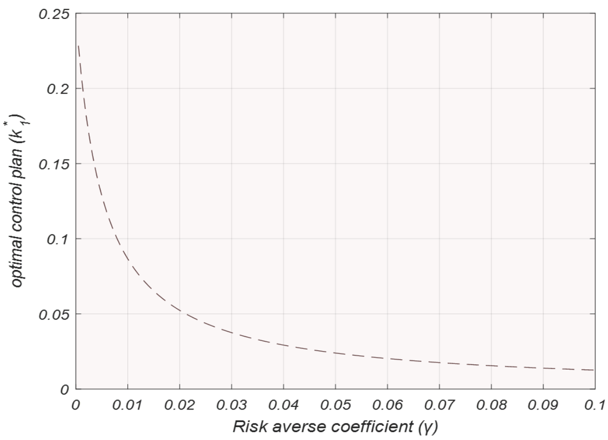

Figure 5: Impact of risk averse coefficient on optimal control plan k_1^*

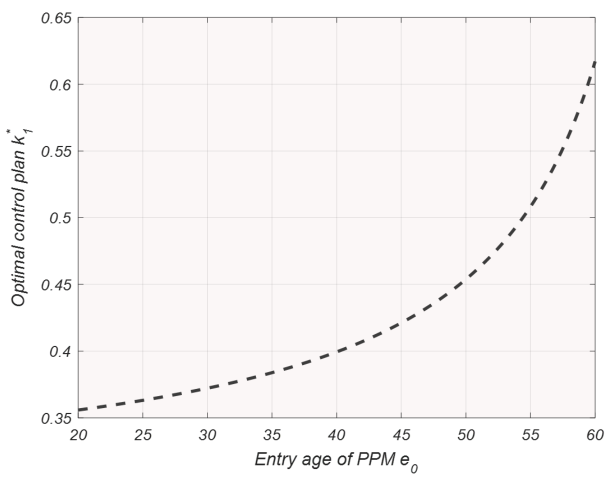

Figure 6: Impact of entry age of PPM on optimal control plan k_1^*

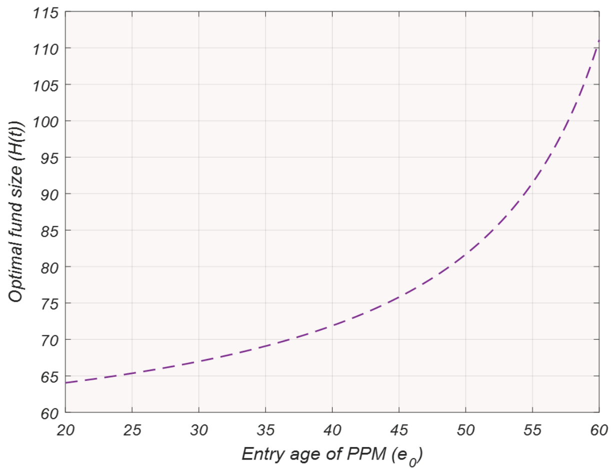

Figure 7: Impact of entry age of PPM on optimal fund size $H(t)$

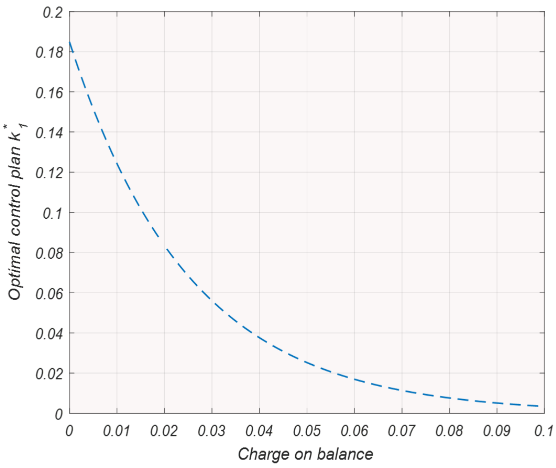

Figure 8: Impact of charge on balance on optimal control plan k_1^*

6 Discussion and conclusion

In this section, the impacts of some parameters on the optimal control plan are discussed. In Figure 1, the graph of optimal control plans for the two risky assets are presented as a function of time; the graph shows that, the proportion of investment in the two risky assets decreases with time and gets smaller and smaller as the PPM approaches retirement age. In Figure 2, the graph of optimal control plan for stock 1 against the optimal fund size is presented; the graph shows that, optimal control plan for the stock 1 decrease as the optimal fund size increases. The implication of this graph is that members with more funds may prefer to invest in less in the risky asset as retirement age approaches and vice versa. Figure 3 is the graph of optimal control plan for the stock 2 against the optimal fund size; the graph shows that, optimal control plan for the stock 2 decreases as the optimal fund size increases. This is similar to figure 2.

In figure 4, the graph of the optimal fund size against the risk free interest rate is presented; the graph shows that the optimal fund size is an increasing function of the risk free interest rate. The implication of the graph is that members with large funds prefers to invest where there is less risk since they may not want to lose what they have gathered already. In figure 5, the graph of optimal control plan against the risk averse coefficient is presented; the graph shows that the optimal control plan is inversely proportional to the risk averse coefficient. This implies that the more fearful the PPM is to investment in risky asset, the more likelihood that he will reduce the proportion of his wealth to be invested in risky asset.

In figure 6, the graph illustrates the impact of PPM's entry age on the optimal control plan. It is observed that the optimal control plan increases as the entry age of the PPM increases. The implication is that PPM with late entry in to the pension scheme, tends to increase the proportion of their wealth to be invested in the risky asset. This can be confirm from figure 7 which shows that the optimal fund size of a PPM is directly proportional to the entry age of the PPM. In figure 8, the relationship between the optimal control plan and the charge on balance is presented. It is observe that the optimal control plan developed by the pension fund administrators (PFA) is a decreasing function of the charge on balance. The implication in figure 7 is that, the higher the charge on balance on investment of the risky asset, the more likelihood that PPM of the scheme will be discouraged from investing a large proportion of their wealth in risky asset and vice versa.

In conclusion, the PPMs portfolios were studied under mortality risk which exhibit constant absolute risk averse. The mortality rate was determined De Moivre mortality force function and a portfolio comprising of two risky assets which forms a 2×2 positive definite matrix was considered to build the PPM's portfolio. The Legendre transformation and dual theory was used to solve for the optimal control plan, value function and the optimal fund size. It was observed that the OCP is a decreasing function of the optimal fund size, risk averse, predetermined interest rate and charge on balance. Also, we observed that the optimal control plan is an increasing function of the PPM's entry age.

Conflict of Interests

The authors declare that there is no conflict of interests regarding this paper.

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