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High order stiffly stable parameter dependent nested hybrid linear multistep methods for stiff ODEs

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Abstract

In this work, we proposed a High Order Stiffly Stable parameter dependent nested hybrid linear multistep method for stiff initial value problems (IVPs) in ordinary differential equations (ODEs). It forms a super class of the linear multistep methods (LMMs), with properties that embedded the characteristics of LMM and hybrid methods. It is hybrid in nature as a result of the incorporation of one or more off-step points and this is aimed at attaining a better stability properties and higher order than the conventional linear multistep methods (LMMs). The proposed method is A- stable and A(α) –stable for step number k ≤ 6. A numerical example is given to demonstrate the application of the methods.

Keywords and phrases: nested hybrid multi-step methods; implicit methods; A-stable; A(α)-stability

1 Introduction

The high order stiffly stable parameter dependent nested hybrid linear multistep method form a super class of linear multistep methods (LMMs), with properties that embedded the characteristics of LMM and hybrid methods. This paper gives a continuous modification of [8], the second derivative continuous linear multistep methods for the solution of stiff initial value problems (IVPs) in ordinary differential equations (ODEs). The motivation lies in the fact that the new formulation offers the advantage of comprehensive solutions to IVPs unlike the discrete solution generated from [3] and [4] methods. The success of these methods is in their attainable stiff stability characteristics useful for resolving the problem posed by stiffness in IVPs [1].

In this regard, an introduction of high order stiffly stable parameter dependent nested hybrid linear multistep method for stiff ODEs is proposed. It is hybrid in nature as a result of the incorporation of one or more off-step points and this is aimed at attaining a better stability properties and higher order than the conventional linear multistep methods (LMM). This type of methods provides a means of bypassing the Dahlquist order and stability barrier of the conventional LMM [3]. The proposed method is A- stable and A(α) –stable for step number k ≤ 6. A numerical example is given to demonstrate the application of the methods.

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According to [9], the general form of linear multistep method (LMMs) is

$$Y_0 = \sum_{j=0}^{k} \alpha_j y_{n+j} + h\beta_k f_{n+k}; \qquad Y_0 = y_{n+c_0}. \tag{1}$$

There has been considerable progress in the development of these methods and its hybrid counterpart [8]

$$Y_{i+1} = \sum_{j=0}^{k} \lambda_j y_{n+j} + h\rho_i f(Y_i); \qquad i = 0(1)s-1; \quad Y_{i+1} = y_{n+c_{i+1}}$$

$$y_{n+k} = \sum_{j=0}^{k-1} \ell_j y_{n+j} + h\eta_{s-1} f(Y_{s-1}) + h\phi_k (f_{n+k} + a f_{n+k-1}); \quad c_i = k - \frac{1}{2^{k-i}}, \quad 0 \le c \le k. \tag{2}$$

where Y_{i+1} and y_{n+k} are the hybrid inputs and output.

Hence, the general form of high order stiffly stable parameter dependent nested hybrid multistep methods is

$$\begin{cases} Y_0 = \sum_{j=0}^{k} \phi_j y_{n+j} + h\lambda_k f_{n+k}; \qquad Y_0 = y_{n+c_0} \\ Y_{i+1} = y_{n+k-1} + \sum_{j=0}^{k} \varphi_j f_{n+j} + h\rho_i f(Y_i); \qquad i = 0(1)s-1; \quad Y_{i+1} = y_{n+c_{i+1}} \\ y_{n+k} = y_{n+k-1} + \sum_{j=0}^{k-2} \delta_j y_{n+j} + h\theta_{s-1} f(Y_{s-1}) + h\gamma_k (f_{n+k} + a f_{n+k-1}); \quad 0 \le c \le k. \end{cases} \tag{3}$$

the parameters ϕ_j, φ_j δ_j, ρ_i, θ_{s-1}, λ_k and γ_k with i = 1, · · · , s-1, j = 0, 1, · · · , k are all real coefficients. The $f_{n+j} = f(x_{n+j}, y_{n+j})$ is the first derivative function, $h = x_{n+1} - x_n$ denotes the mesh size, k and s represents the step number, and the stage respectively, while $Y_i = y(x_n + c_i h) + Oh^{q_{i+1}}$, i = 0, · · ·, s-1, and y_{n+k} are the stage and the output point of the methods in (3). The $f(Y_i) = \{f(x_n + c_i h, Y_i)\}_{i=0}^{s-1}$ denotes the derivative of the stages. The stability region of the formulae in (3) depends on the choice of the parameter "a". The $c = [c_0, c_1, \cdots, c_s]^T$ is the abscissa vector of the input methods. The elements in c are computed from the abscissa generator: $c_i = \left\{k - \frac{1}{2^{k-i}}\right\}_{i=0}^{s-1}$, k = 1, 2… s-1.

Interestingly, many cases of these nested methods are known to be stiffly stable for the numerical solution of stiff initial value problems in ODEs, [10],

$$y' = f(x, y),\ x \in [x_0, X],\ y(x_0) = y_0,\ f : \Re \times \Re^m \to \Re^m. \tag{4}$$

In ordinary differential equations where $y(x)$ and $f(x,y)$ may be vectors. We seek the numerical solution of this problem using parameter dependent nested hybrid multistep methods (PDNHLMMs) and its hybrid counterpart obtained by the transformation of the continuous linear multistep method of [11]. Such formulae are the adaptive nested Runge - Kutta (ANRK) methods [9], hybrid multistep formulae with nested hybrid predictors [5], second derivative linear multistep with nested predictors [6] which was introduced to overcome the limitation of order reduction that is often encountered during the implementation process in some diagonally and fully implicit Runge-Kutta methods and also to overcome the limitation encountered by nested method in the area of long execution time when applied to large scale initial value problems. These formulae developed did very well in extensive numerical computations of stiff ODEs. The interest of this paper is to introduce an High Stiffly Stable parameter dependent nested hybrid linear multistep method for the solution of the IVPs (1).

2 Order conditions

By Taylor's series technique, the local truncation error of the stage method and the output method in (3) at x_n gives the following order conditions:

$$h^{q_0}:\begin{cases}\sum\limits_{j=0}^{k}\phi_j=1, & q_0=0,\\ \sum\limits_{j=1}^{k}j\phi_j+\lambda_k=c_0, & q_0=1,\\ \dfrac{1}{q_0!}\sum\limits_{j=1}^{k}j^{q_0}\phi_j+\dfrac{1}{(q_0-1)!}k^{q_0-1}\lambda_k=\dfrac{c_0^{q_0}}{q_0\,!}; & q_0=2,3,\ldots,\end{cases} \tag{5a}$$

The error constant of the method in (5a) is

$$C_{q_0+1}=\frac{1}{(q_0+1)!}\sum_{j=0}^{k}j^{q_0+1}\phi_j+\frac{1}{q_0\,!}k^{q_0}\lambda_k=\frac{c_i^{q_0+1}}{(q_0+1)\,!}+\mathrm{O}(\mathrm{h}^{q_0+1}). \tag{5b}$$

Equations (5a) and (5b) are the order conditions and error constants of the first input method in (3) while the order conditions and error constants of the second input method in (3) are

$$h^{q_i}:\begin{cases}\sum\limits_{j=0}^{k}\varphi_j+\rho_0=4+c_1, & i=1,2,\ldots,s,\quad i=0(1)s-1,\\ \sum\limits_{j=1}^{k}j\varphi_j+c_0\rho_i=8-\dfrac{c_i^2}{2!}, & q_i=1,\\ \dfrac{1}{q_i\,!}\sum\limits_{j=1}^{k}j^{q_i}\varphi_j+\rho_0c_0^{q_i-1}=-\dfrac{c_0^{q_i}}{q_i!}+\dfrac{64}{q_i\;!}; & q_i=2,3,\ldots,\end{cases}$$

(6a)

The error constant of the method in (3) is

$$C_{q_i+1}=\frac{1}{(q_i+1)!}\sum_{j=0}^{k-1}j^{q_i+1}\varphi_j+\rho_0c_0^{q_i}=\frac{c_i^{q_i+1}}{(q_i+1)\,!}+\frac{64}{(q_i+1)!}\mathrm{O}(\mathrm{h}^{q_i+1}).$$

(6b)

For the output method in (3), the order conditions and error constants are

$$h^{p}:\begin{cases}\sum\limits_{j=0}^{k-1}\delta_j+\gamma_k=1, & p=0,\\ \sum\limits_{j=1}^{k-1-}j\,\delta_j+\gamma_k(5+a)=c_4, & p=1,\\ \dfrac{1}{p!}\sum\limits_{j=1}^{k-1}j^{p}\delta_j+\dfrac{1}{(p-1)!}\theta_{s-1}c_{s-1}^{p-1}+\dfrac{1}{(p-1)!}\left(a(-1+k)^{p-1}+k^{p-1}\right)\gamma_k=\dfrac{k^{p}}{p\;!}; & p=2,3,\ldots,\end{cases}$$

(7a)

and

$$C_{p+1}=\frac{1}{(p+1)!}\sum_{j=1}^{k-1}j^{p+1}\delta_j+\frac{1}{p\;!}\theta_{s-1}c_{s-1}^{p}+\frac{1}{p\;!}\left(a(-1+k)^{p}+k^{p}\right)\gamma_k=\frac{k^{p+1}}{(p+1)\,!}+\mathrm{O}(\mathrm{h}^{p+1})$$

(7b)

Derivation of the multistep nested hybrid methods

$k=1\ s=1\ p=q+1=3\ c_0=\frac{1}{2}\ a=\frac{1}{2}$			
Y_0	$\frac{1}{4}(y_n+3y_{n+1})-\frac{h}{4}f_{n+1};$	$q_0=2,$	$C_3=\frac{1}{48}$
y_{n+1}	$y_{n+1}=y_n+hf(Y_0)$	$p=2,$	$C_3=\frac{1}{24}$
$k=2\ s=2\ p=q_1=q_0+1\ c_0=\frac{7}{4}\ c_1=\frac{3}{2}\ a=\frac{1}{4}$			
Y_0	$\frac{1}{256}(-3y_n+28y_{n+1}+231y_{n+2})-\frac{21h}{128}f_{n+2};$	$C_4=\frac{7}{2048},$	$q_0=3$
Y_1	$y_{n+1}+\frac{1}{96}(-hf_n+30hf_{n+1}-13hf_{n+2}+32h\,f(Y_0));$	$C_5=\frac{179}{92160},$	$q_1=4$
y_{n+2}	$y_{n+2}=-\frac{1}{23}y_n+\frac{24}{23}y_{n+1}-\frac{56}{69}hf(Y_1)+\frac{8}{69}h(f_{n+2}+af_{n+1})$	$C_4=\frac{1}{84},$	$p=3.$
$k=3\ s=3\ p=q_2=q_1=q_0+1=5\ c_0=\frac{23}{8}\ c_1=\frac{11}{4}\ c_2=\frac{5}{2}\ a=\frac{1}{2}$			
Y_0	$\frac{1}{49152}(70y_n-48328y_{n+1}+2070y_{n+2}+47495y_{n+3}-4830hf_{n+3});\ \ q_0=4,$	$C_5=\frac{161}{262144}$	$q_0=4,$
Y_1	$y_{n+2}+\frac{1143}{10496}hf_n-\frac{1983}{51200}hf_{n+1}+\frac{30549}{71680}hf_{n+2}-\frac{4977}{10240}hf_{n+3}+\frac{3396h}{824025}f(Y_0);$	$C_6=\frac{681}{409600},$	$q_1=5,$
Y_2	$y_{n+2}+\frac{179}{63360}hf_n-\frac{319}{13440}hf_{n+1}+\frac{1979}{5760}hf_{n+2}-\frac{601}{5760}hf_{n+3}+\frac{976h}{3465}f(Y_1);$	$C_6=\frac{41}{46080},$	$q_2=5,$
y_{n+3}	$y_{n+2}-\frac{1}{150}hf_n-\frac{1}{30}hf_{n+1}+\frac{58}{75}hf(Y_2)-\frac{2}{15}h(f_{n+3}+af_{n+2});$	$C_5=\frac{11}{2880},$	$p=4,$

Using the general form of the k-step High Order Stiffly Stable parameter dependent nested hybrid linear multistep methods in (3) above, the method for $k_{i's}$ was derived by applying Taylor's series scheme to the expansion, by fixing $k_{i's}$ for every i = 1(n)6, into (5), (6) and (7) respectively, we have by setting

$k=4\ s=4\ p=q_3=q_2=q_1=q_0+1=6\ c_0=\frac{63}{16}\ c_1=\frac{23}{8}\ c_2=\frac{15}{4}\ c_3=\frac{7}{2}\ a=\frac{1}{2}$			
Y_0	$-\frac{7.3\times10^3}{3.4\times10^7}y_n+\frac{3.3\times10^3}{2.1\times10^6}y_{n+1}-\frac{4.4\times10^4}{8.4\times10^6}y_{n+2}+\frac{3.1\times10^4}{2.1\times10^6}y_{n+3}+\frac{3.3\times10^7}{3.3\times10^7}y_{n+4}-\frac{4.6\times10^5}{8.4\times10^6}hf_{n+4};$	$C_6=\frac{3.1\times10^4}{2.7\times10^8},$	$q_0=5,$
Y_1	$y_{n+3}-\frac{1.3\times10^5}{5.3\times10^7}hf_n+\frac{6.7\times10^5}{3.4\times10^7}hf_{n+1}-\frac{2.3\times10^6}{3.0\times10^7}hf_{n+2}+\frac{2.7\times10^4}{5.5\times10^4}hf_{n+3}-\frac{6.8\times10^6}{5.8\times10^6}hf_{n+4}+\frac{9.4\times10^5}{5.9\times10^5}hf(Y_0);$	$C_7=\frac{2.0\times10^7}{1.8\times10^{10}},$	$q_1=6,$
Y_2	$y_{n+3}-\frac{6.8\times10^2}{3.2\times10^5}hf_n+\frac{3.9\times10^3}{2.3\times10^5}hf_{n+1}-\frac{6.7\times10^2}{1.0\times10^4}hf_{n+2}+\frac{3.3\times10^4}{7.2\times10^4}hf_{n+3}-\frac{5.4\times10^2}{1.3\times10^3}hf_{n+4}+\frac{1.9\times10^4}{2.5\times10^4}hf(Y_1);$	$C_7=-\frac{3.5\times10^4}{3.7\times10^7},$	$q_2=6,$
Y_3	$y_{n+3}-\frac{41}{34560}hf_n+\frac{589}{63360}hf_{n+1}-\frac{131}{3360}hf_{n+2}+\frac{6347}{17280}hf_{n+3}-\frac{997}{11520}hf_{n+4}+\frac{520}{2079}hf(Y_2);$	$C_7=-\frac{3.5\times10^{-4}}{3.7\times10^{-7}},$	$q_3=6,$
y_{n+4}	$y_{n+3}+\frac{11}{3368}hf_n+\frac{448}{1575}hf_{n+1}+\frac{234}{315}hf_{n+2}+\frac{8328}{11025}hf(Y_3)+\frac{179}{1260}h\left(f_{n+4}+a\,f_{n+3}\right)$	$C_6=\frac{32}{35151},$	$p=5$
$k=5\ s=5\ p=q_4=q_3=q_2=q_1=q_0+1=7\ c_0=\frac{159}{32}\ c_1=\frac{79}{16}\ c_2=\frac{39}{8}\ c_3=\frac{19}{4}\ c_4=\frac{9}{2}\ a=\frac{1}{2}$			
Y_0	$\frac{1.6\times10^6}{4.2\times10^{10}}y_n-\frac{9.8\times10^6}{3.4\times10^{10}}y_{n+1}+\frac{4.4\times10^6}{4.3\times10^9}y_{n+2}-\frac{2.0\times10^7}{8.6\times10^9}y_{n+3}+\frac{4.1\times10^7}{8.6\times10^9}y_{n+4}+\frac{1.7\times10^{11}}{1.7\times10^{11}}y_{n+5}-\frac{2.5\times10^8}{8.6\times10^9}hf_{n+5};$	$C_7=\frac{1.2\times10^7}{5.5\times10^{11}}$	$q_0=6$
Y_1	$y_{n+4}+\frac{1.5\times10^8}{9.9\times10^{10}}hf_n-\frac{2.9\times10^9}{2.4\times10^{11}}hf_{n+1}+\frac{8.3\times10^8}{1.8\times10^{10}}hf_{n+2}-\frac{2.4\times10^9}{2.0\times10^{10}}hf_{n+3}+\frac{3.2\times10^{10}}{5.8\times10^{10}}hf_{n+4}-\frac{4.5\times10^9}{1.9\times10^9}hf_{n+5}+\frac{1.6\times10^9}{5.8\times10^8}hf(Y_0);$	$C_7=\frac{2.2\times10^{10}}{2.7\times10^{13}}$	$q_1=7$

Y_2	$y_{n+4}+\frac{2.1\times10^{7}}{1.5\times10^{10}}hf_n-\frac{1.9\times10^{7}}{1.7\times10^{9}}hf_{n+1}+\frac{1.9\times10^{8}}{4.4\times10^{9}}hf_{n+2}-\frac{3.3\times10^{8}}{2.9\times10^{9}}hf_{n+3}+\frac{3.0\times10^{8}}{5.7\times10^{8}}hf_{n+4}$ $-\frac{1.9\times10^{8}}{1.9\times10^{8}}hf_{n+5}+\frac{6.8\times10^{7}}{4.7\times10^{7}}hf(Y_1);$	$C_8=\frac{2.2\times10^{8}}{2.9\times10^{11}}$	$q_2=7$
Y_3	$y_{n+4}+\frac{3.5\times10^{4}}{3.0\times10^{7}}hf_n-\frac{6.7\times10^{5}}{7.1\times10^{7}}hf_{n+1}+\frac{9.5\times10^{5}}{2.6\times10^{7}}hf_{n+2}-\frac{2.2\times10^{4}}{2.3\times10^{5}}hf_{n+3}+\frac{8.0\times10^{6}}{1.6\times10^{7}}hf_{n+4}$ $-\frac{8.6\times10^{5}}{2.3\times10^{6}}hf_{n+5}+\frac{1.6\times10^{6}}{2.3\times10^{6}}hf(Y_2);$	$C_8=\frac{1.2\times10^{6}}{2.1\times10^{9}}$	$q_3=7,$
Y_4	$y_{n+4}+\frac{7.6\times10^{2}}{1.2\times10^{6}}hf_n-\frac{4.9\times10^{3}}{9.8\times10^{5}}hf_{n+1}+\frac{3.5\times10^{4}}{1.8\times10^{6}}hf_{n+2}-\frac{6.3\times10^{4}}{1.1\times10^{6}}hf_{n+3}$ $+\frac{3.7\times10^{5}}{9.7\times10^{5}}hf_{n+4}-\frac{2.4\times10^{4}}{3.2\times10^{5}}hf_{n+5}+\frac{6.3\times10^{4}}{2.8\times10^{5}}hf(Y_3)$	$C_7=\frac{1.3\times10^{4}}{4.3\times10^{7}}$	$q_4=7$
y_{n+5}	$y_{n+4}+\frac{12}{7290}hf_n+\frac{83}{7560}hf_{n+1}+\frac{35}{1080}hf_{n+2}+(\frac{589}{1080}-\frac{79}{162})hf_{n+3}+\frac{19032}{25515}hf(Y_4)$ $-\frac{79}{540}h\left(f_{n+5}+a\,f_{n+4}\right);$	$C_7=\frac{373}{362880},$	$p=6$
$k=6\,,s=6\,,\ p=q_5=q_4=q_3=q_2=q_1=q_0+1=8,\ c_0=\frac{383}{64},\ c_1=\frac{191}{32},c_2=\frac{95}{16},c_3=\frac{47}{8},c_4=\frac{23}{4},\ c_5=\frac{11}{2},\ a=\frac{1}{4},$			
Y_0	$-\frac{9.2\times10^{8}}{1.4\times10^{13}}y_n+\frac{9.9\times10^{9}}{1.7\times10^{14}}y_{n+1}-\frac{6.2\times10^{10}}{2.8\times10^{14}}y_{n+2}+\frac{9.2\times10^{9}}{1.8\times10^{13}}y_{n+3}-\frac{1.2\times10^{11}}{1.4\times10^{14}}y_{n+4}$ $+\frac{5.0\times10^{10}}{3.5\times10^{13}}y_{n+5}-\frac{1.4\times10^{15}}{1.4\times10^{15}}y_{n+6}-\frac{1.1\times10^{12}}{7.0\times10^{13}}hf_{n+6};$	$C_8=\frac{1.5\times10^{11}}{3.6\times10^{16}}$	$q_0=7$

,

Y_1	$y_{n+5}-\frac{1.6\times10^{14}}{1.6\times10^{17}}hf_n+\frac{6.3\times10^{13}}{7.4\times10^{15}}hf_{n+1}-\frac{1.4\times10^{14}}{3.9\times10^{15}}hf_{n+2}+\frac{8.6\times10^{15}}{9.9\times10^{16}}hf_{n+3}-\frac{4.9\times10^{15}}{2.9\times10^{16}}hf_{n+4}$ $+\frac{2.3\times10^{13}}{3.9\times10^{13}}hf_{n+5}-\frac{9.9\times10^{15}}{2.1\times10^{15}}hf_{n+6}-\frac{1.1\times10^{13}}{2.1\times10^{12}}hf(Y_0);$	$C_9=\frac{5.8\times10^{17}}{1.0\times10^{21}}$	$q_1=8$
Y_2	$y_{n+5}-\frac{2.2\times10^{10}}{2.3\times10^{13}}hf_n+\frac{2.6\times10^{10}}{3.2\times10^{12}}hf_{n+1}-\frac{5.1\times10^{11}}{1.5\times10^{13}}hf_{n+2}+\frac{4.8\times10^{10}}{5.7\times10^{11}}hf_{n+3}$ $-\frac{8.3\times10^{9}}{5.2\times10^{10}}hf_{n+4}+\frac{1.1\times10^{12}}{1.9\times10^{12}}hf_{n+5}-\frac{2.7\times10^{11}}{1.2\times10^{11}}hf_{n+6}+\frac{4.3\times10^{10}}{1.6\times10^{10}}hf(Y_1);$	$C_9=\frac{5.4\times10^{11}}{9.9\times10^{14}}$	$q_2=8$
Y_3	$y_{n+5}-\frac{7.8\times10^{7}}{9.1\times10^{10}}hf_n+\frac{6.1\times10^{8}}{8.0\times10^{10}}hf_{n+1}-\frac{6.2\times10^{7}}{2.0\times10^{9}}hf_{n+2}+\frac{1.7\times10^{10}}{2.1\times10^{11}}hf_{n+3}-\frac{9.5\times10^{9}}{6.2\times10^{10}}hf_{n+4}$ $+\frac{2.8\times10^{9}}{5.0\times10^{9}}hf_{n+5}-\frac{1.7\times10^{10}}{1.8\times10^{10}}hf_{n+6}+\frac{1.2\times10^{7}}{8.6\times10^{6}}hf(Y_2);$	$C_9=\frac{2.8\times10^{11}}{5.6\times10^{14}}$	$q_3=8$
Y_4	$y_{n+5}-\frac{1.2\times10^{6}}{1.7\times10^{9}}hf_n+\frac{1.5\times10^{6}}{2.4\times10^{8}}hf_{n+1}-\frac{2.8\times10^{7}}{1.1\times10^{9}}hf_{n+2}+\frac{1.3\times10^{7}}{2.1\times10^{8}}hf_{n+3}-\frac{4.7\times10^{6}}{3.7\times10^{7}}hf_{n+4}$ $+\frac{1.9\times10^{6}}{3.7\times10^{7}}hf_{n+5}-\frac{2.5\times10^{6}}{7.3\times10^{6}}hf_{n+6}+\frac{9.9\times10^{6}}{1.5\times10^{7}}hf(Y_3);$	$C_9=\frac{3.7\times10^{7}}{9.4\times10^{10}}$	$q_4=8$
Y_5	$y_{n+5}-\frac{1.3\times10^{4}}{3.6\times10^{7}}hf_n+\frac{5.4\times10^{3}}{1.6\times10^{6}}hf_{n+1}-\frac{1.2\times10^{4}}{8.6\times10^{5}}hf_{n+2}+\frac{6.8\times10^{4}}{1.9\times10^{6}}hf_{n+3}-\frac{6.3\times10^{4}}{8.6\times10^{5}}hf_{n+4}+\frac{1.7\times10^{5}}{4.5\times10^{5}}hf_{n+5}$ $-\frac{5.1\times10^{5}}{7.7\times10^{6}}hf_{n+6}+\frac{6.3\times10^{4}}{2.8\times10^{5}}hf(Y_4)$	$C_9=\frac{3.0\times10^{6}}{1.5\times10^{10}}$	$q_5=8$
y_{n+6}	$y_{n+5}-\frac{1279}{850080}hf_n-\frac{23597}{2086560}hf_{n+1}+\frac{815}{90160}hf_{n+2}-\frac{1079}{38640}hf_{n+3}+(\frac{24487}{30240}+\frac{3660}{5152})hf_{n+4}$ $+\frac{3861776}{5020785}hf(Y_5)+\frac{732}{5152}h\left(f_{n+6}+a\ f_{n+5}\right);$	$C_8=\frac{-1177}{947520}$	P=7

The stability region of the formula in (3) depends on the choice of the value of the parameter “a” which has the function of smoothening the method for better stability properties. It is obvious that during the application of the methods for the numerical integration of stiff IVPs, errors are bound to occur at some stages of the computation due to inaccuracy inherent in the formula and the arithmetic operations adopted during the computer implementation. The magnitude of the error determines the degree of accuracy and stability of the method [2]. The minimum properties a numerical method could possess are stability properties [14].

Some of the advantages of this work are that It is of high order and stiffly stable and it has a better stability function. The better stability function of the method was made possible because of the present of a parameter “a” which lies between $0\ and\ 1$and it help to obtain a better stability function thereby leading to a smooth stability curve.

Applying the method in (3) to the test equation $y^{/} = \lambda y,\ \mathrm{Re}(\lambda) < 0,$ gives the stability polynomial

$$\pi_s(r,z) = w^k - w^{k-1} - z\sum_{j=0}^{k-2}\delta_j w^j - z\theta_{s-1}(R_s(w,z)) - z\gamma_s\left(w^k + a\,w^{k-1}\right), \quad z = \lambda h,$$

(8)

where

$$R_s(w,z) = w^{k-1} + z\sum_{j=0}^{k}\varphi_j w^j + z\rho_{s-2}\left(R_{s-1}(w,z)\left(R_{s-2}(w,z)\left(R_{s-3}(w,z)\left(R_{s-4}(w,z)\left(R_{s-5}(w,z)\right)\right)\right)\right)\right).$$

(9)

The behavior of the numerical solution of the High Order stiffly Stable parametric dependent nested hybrid linear multistep methods will depend on the resulting roots $(w_j(z))$ of $\pi(r,z)=0$. Adopting the definitions in [13], we have:

$R_0 = \left\{ \pi(w,0) = 0 \middle| \max_{1\le j\le k} \left|w_j(z)\right| \le 1,\ j = 1, 2, \ldots, k \right\}$ is called the zero-stability of the methods in (3).

$R_a = \left\{ \pi(w,z) = 0 \middle| \max_{1\le j\le k} \left|w_j(z)\right| \le 1,\ j = 1, 2, \ldots, k \right\}$ is called the absolute stability of the method in (3).

The methods in (3) with stability polynomial $\pi(w,z)$is A(α)-stable if $|\ \pi(w,z)\ | \le 1$ for all z∈S(α) , α∈ (0, π) , where α is an angle of absolute stability.

$R_A = \left\{ \left|\pi(w,z)\right| \le 1;\ R(z) \le 0 \right\}$is called A-stability of the method (3).

The stability plots of the methods in (3) for $k \le 6$are given in Figure 1.

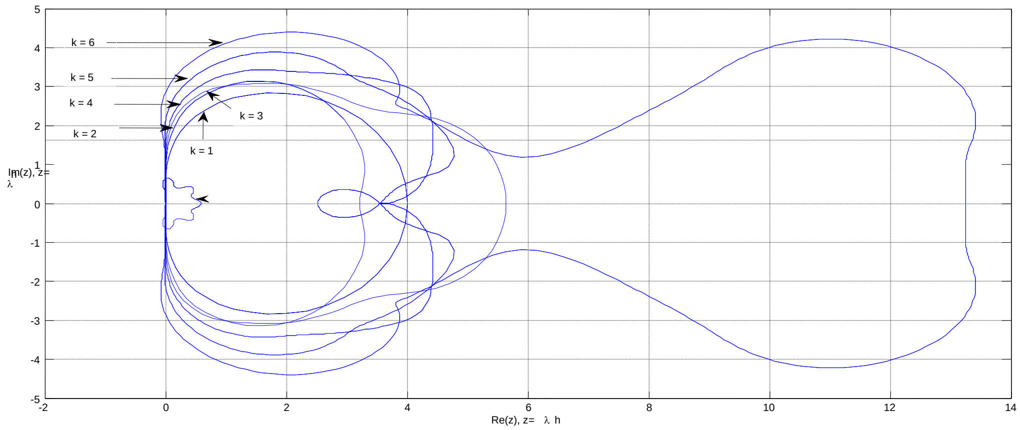

Figure 1: The stability regions (exterior of the closed curves) for the k-step PDNHM method in (2); $k \leq 6$.

3 Numerical experiments

In this section, we consider the application of the proposed method in scheme for k=1

$$Y_0 = \frac{1}{4}(y_n + 3y_{n+1}) - \frac{h}{4} f_{n+1}; \qquad q_0 = 2, \quad C_3 = \frac{1}{48},$$

$$y_{n+1} = y_n + hf(Y_0); \; p = 2, \quad C_3 = \frac{1}{24}.$$

(10)

which is of order 3, and also independently derived using Taylor's series expansion on the initial value problem below;

$$y' = \begin{bmatrix} -0.1 & 0 & 0 & 0 \\ 0 & -10 & 0 & 0 \\ 0 & 0 & -100 & 0 \\ 0 & 0 & 0 & -1000 \end{bmatrix} y, \; y(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, h = 0.001,$$

$x = [0, \; 1], \;\; y = [y_1, y_2, y_3, y_4]^T$

$$\textit{Exact solution}: y(x) = [\exp^{-0.1x}, \exp^{-10x}, \exp^{-100x}, \exp^{-1000x}]^T$$

with x = 0(0.1)2000. This problem is stiff. Since S = 10000, where S is the stiffness ratio.

In solving the IVPs above, the implicitness in the methods has to be resolved using the Newton Raphson iterative scheme below as suggested by [5] and [6],

$$y_{n+k}^{[s+1]} = y_{n+k}^{[s]} - (F'(y_{n+k}^{[s]}))^{-1} F(y_{n+k}^{[s]}), \quad s = 0, 1, \ldots$$

(11)

where $F'(y_{n+k}^{[s]})$ is the Jacobian matrix. While an explicit one-step method

$$y_{n+1}^{[0]} = y_n + \frac{h}{2}\left(f_{n-1} + f_n\right).$$

(12)

is used to generate the starting value for the schemes in (10).

Table 1: The Numerical Solution of the Problem on PDNHLMM compared with A(α)-Stable LMM

x	PDNHLMM y_n	A(α)-Stable LMM y_n	Exact Solution $y(x)$	PDNHLMM Error = $\lvert y_{(x)} - y_n \rvert$	A(α)-Stable LMM Error = $\lvert y_{(x)} - y_n \rvert$
0.2	$2.2569438502\, e \times 10^{-9}$	2.1190578206	$2.2779270412\, e \times 10^{-9}$	$2.0983190958\, e \times 10^{-11}$	$1.9920072216\, e \times 10^{-16}$
0.4	$5.0959769196\, e \times 10^{-18}$	5.1315793862	$4.6951575726\, e \times 10^{-18}$	$4.0081934701\, e \times 10^{-19}$	$5.0021074282\, e \times 10^{-16}$
0.6		1.1511722549	$9.6774410387\, e \times 10^{-27}$	$1.8288180836\, e \times 10^{-27}$	$9.6663711108\, e \times 10^{-16}$
0.8	$2.5980101770\, e \times 10^{-35}$	2.5960009977	$1.9946692652\, e \times 10^{-35}$	$6.0334091172\, e \times 10^{-36}$	$4.7122272259\, e \times 10^{-16}$
1.0	$5.8660741151\, e \times 10^{-44}$	5.8571556981	$4.1113197817\, e \times 10^{-44}$	$1.7547543333\, e \times 10^{-44}$	$3.9931006410\, e \times 10^{-16}$

The graphical representation of the results from the problem is shown in Figure 2 below.

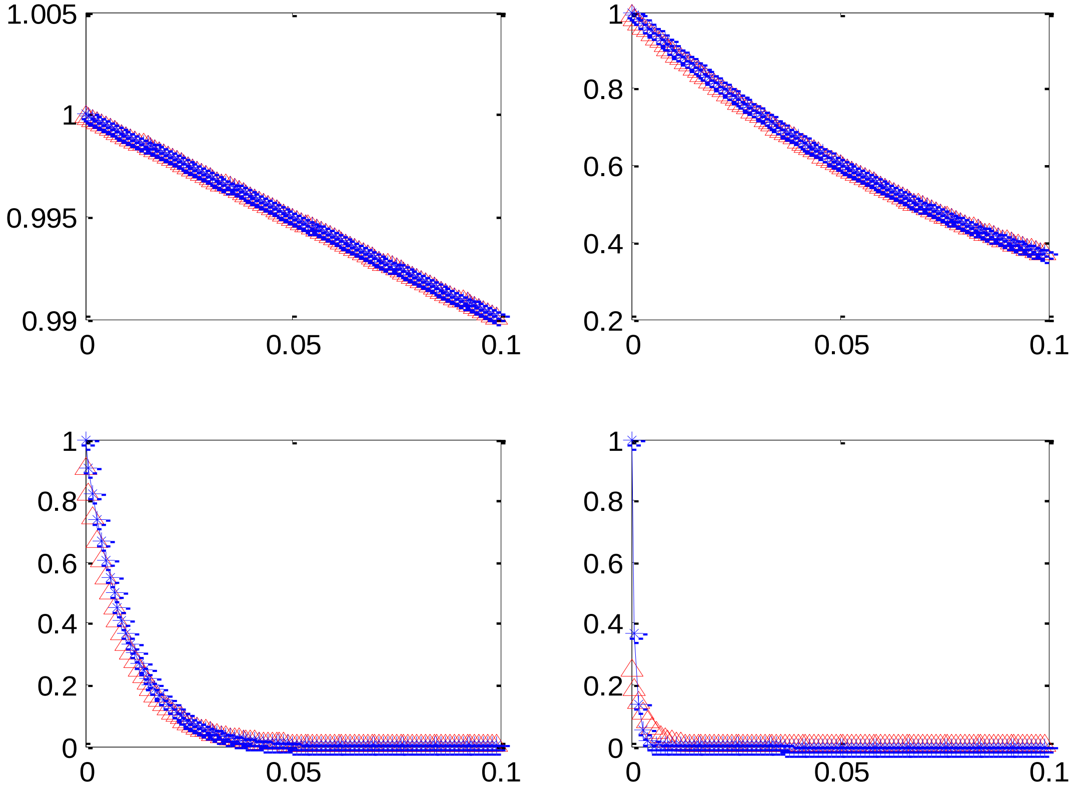

Figure 2

3.1 Discussion of results

Table 1: A careful look at the results from the new methods on the problem solved as compared with the results A(α)-Stable LMM with the step-size h = 0.001used for the problem for xε{0,1}. It reveals that the new methods on the problem solved have a very low error level as compared to the A(α)-Stable LMM. This shows that the new methods (PDNHLMMs) are much more accurate, effective and efficient when solving stiff IVPs in ODEs.

Figure 2: Graphical Representation (Blue Represent Numerical Solutions while Red represents Exact Solutions).Unlike the Euler and RKM, the consistency and convergence of our method can be ascertained during the implementation process. A linear multistep is consistence if the numerical solution converges to that of the exact solution and also, if the method is able to solve exactly two or more ODE problems with different initial conditions. A linear multistep is convergent if and only if it is stable and consistent. A careful look at the graphs of our methods shows that our methods are consistence and convergence as the numerical solutions converges to that of the exact solutions This is a proof that our methods (PDNHLMMs) compared favorably well with other existing methods.

4 Conclusion

This paper proposes and studies the high Order Stiffly Stable parameter dependent nested hybrid linear multistep methods for the numerical solution of the IVPs (1). We derived the order conditions for methods via Taylor's series expansion; based on the order conditions presented in this paper, hybrid methods for step number $k \leq 6$ were derived. The results of the numerical experiments confirm that the results of our new methods are more accurate than the high quality methods in the scientific literature.

Following this work, there are several interesting modifications to the new methods in (3). For example, a classical modification of the methods can leads to the well-known GLMs [11] of higher order than the existing methods in the literature.

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