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Convergence analysis of an inertial algorithm for generalized mixed equilibrium problems and Bregman quasi asymptotically nonexpansive mappings

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Abstract

The aim of this article is to introduce an inertial algorithm for approximating solution of a system of generalized mixed equilibrium problem and common fixed point of countable families of Bregman quasi asymptotically nonexpansive mappings. The strong convergence theorem has been establish in reflexive Banach spaces. Our results are significant improvement of some recent results announced in the literature.

Keywords and phrases: strong convergence; generalized mixed equilibrium problem; Bregman projections; Bregman quasi asymptotically nonexpansive mappings.

1 Introduction

An inertial-type algorithm is a techniques for accelerating the convergence of the sequence of an algorithm introduced by Polyak [27]. Many problems have been solved using inertial algorithms (for more details see, [11, 13, 16, 18, 24] and the references therein).

Let E be a reflexive Banach space with E^* as the dual space of E and C be a nonempty closed convex subset of E. Let $\Theta : C \times C \longrightarrow \mathbb{R}$ be a bifunctions, where $\mathbb{R}$ is the set of real numbers, $\Psi : C \longrightarrow E^*$ is a nonlinear continuous monotone mapping and $\varphi : C \longrightarrow \mathbb{R}$ be a convex and lower semi continuous function. The short form $GMEP$ is consider as generalized mixed equilibrium problem [19] mean that to find $x \in C$ such that:

$$\Theta(x, y) + \varphi(y) - \varphi(x) + \langle \Psi x, y - x\rangle \geq 0, \forall y \in C. \tag{1}$$

Also the short form $GMEP(\Theta, \Psi, \varphi)$ is consider as the set of solutions of generalized mixed equilibrium problem (1) is defined by

$$GMEP(\Theta, \Psi, \varphi) = \{x \in C : \Theta(x, y) + \varphi(y) - \varphi(x) + \langle \Psi x, y - x\rangle \geq 0, \forall y \in C\}.$$

Numerous problems in same areas such as nonlinear analysis, optimization, economics and variational inequality problem can be reduced to finding solutions of some equilibrium problems (see [12, 14] and the references therein). Moreover, various methods have been studied for solutions of some equilibrium problems in Hilbert spaces, (see [12, 17, 32, 36] and the references therein).

In 1967, Bregman [5] introduced an effective technique by considering the Bregman distance function D_f for designing and analyzing feasibility and optimization algorithms. The Bregman technique has applied for approximating feasibility problems, optimization problems and fixed point problem for

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nonlinear mappings (see, e.g [3, 15, 25, 34] and the references therein).

A mapping $J : E \to 2^{E^*}$ defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \forall x \in E,$$

is consider as normalized duality mapping on E and $\langle x, x^* \rangle$ present as the pairing between element of E and that of E^*.

Let E be a reflexive Banach space, $f : E \to (-\infty, +\infty]$ is a proper, lower semi-continuous and convex function. dom$f := \{x \in E : f(x) < +\infty\}$, is considered as the domain of f. Let $x \in$ int(dom$f$); the subdifferential of f at x is the convex set defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle x^*, y - x \rangle \leq f(y),\ \forall y \in E\}.$$

The Fenchel conjugate of f is the function $f^* : E^* \to (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{\langle x^*, x \rangle - f(x) : x \in E\}.$$

The Young - Fenchel inequality present as:

$$\langle x^*, x \rangle \leq f(x) + f^*(x^*), \forall x \in E, x^* \in E^*.$$

A function f on E is coercive [22] if the sublevel set of f is bounded :

$$\lim_{\|x\| \to +\infty} f(x) = +\infty.$$

A function f on E is said to be strongly coercive [35] provided that

$$\lim_{\|x\| \to +\infty} \frac{f(x)}{\|x\|} = +\infty.$$

The right-hand derivative of f at x in the direction y, for any $x \in intdomf$ and $y \in E$ is defined by

$$f^0(x, y) := \lim_{t \to 0^+} \frac{f(x + ty) - f(x)}{t}.$$

If $\lim_{t\to 0} \frac{f(x+ty)-f(x)}{t}$ exists for any y, then the function f is called Gâteaux differentiable at x. Therefore, $f^0(x, y)$ coincides with $\nabla f(x)$, the value of the gradient of f at x. The function f is said to be Gâteaux differentiable if it is Gâteaux differentiable for any $x \in$ int(dom$f$). The function f is said to be Fréchet differentiable at x if this limit is attained uniformly in y with $\|y\| = 1$. Also f is said to be uniformly Fréchet differentiable on a subset C of E if the limit is attained uniformly for $x \in C$ and $\|y\| = 1$. Recall that if f is Gâteaux differentiable (resp. Fréchet differentiable) on int(dom$f$), then f is continuous and its Gâteaux derivative ∇f is norm-to-weak* continuous (resp. norm-to-norm continuous) on int(dom(f)) (see [4, 7]).

Definition 1.1. [6] The function f is said to be:

(1) Essentially smooth, if ∂f is both locally bounded and single-valued on its domain;

(ii) Essentially strictly convex, if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every subset of dom$f$;

(iii) Legendre, if it is both essentially smooth and essentially strictly convex.

Remark 1.2. Note that the following results follows, if E is a reflexive Banach space:

(i) f is essentially smooth if and only if f^* is essentially strictly convex (see [6], Theorem 5.4).

(ii) $(\partial f)^{-1} = \partial f^*$ (see [7]).

(iii) f is Legendre if and only if f^* is Legendre (see [6],Corollary 5.5).

(iv) If f is Legendre, then ∇f is a bijection satisfying $\nabla f = (\nabla f^*)^{-1}$, ran$\nabla f$ = dom$\nabla(f^*)$ = int(domf^*) and ran ∇f^* = dom$f$ = int(dom$f$) (see [6], Theorem 5.10), where ran stands for the range.

Consider $f : E \longrightarrow (-\infty, +\infty]$ as a convex and Gâteaux differentiable function. The function D_f : $dom \times intdomf \longrightarrow [-\infty, +\infty)$ defined by

$$D_f(y, x) := f(y) - f(x) - \langle \nabla f(x), y - x\rangle$$

is called the Bregman distance with respect to f. Further the following property [30] is from the definition of D_f:

$$D_f(z, x) := D_f(z, y) + D_f(y, x) + \langle \nabla f(y) - \nabla f(x), z - y\rangle$$

Definition 1.3. [37] Let $T : C \longrightarrow C$ with $F(T) \neq \emptyset$ be a mapping and $F(T) = \{x \in C : Tx = x\}$ is a set of fixed points of T . Then
(1) A point $p \in C$ is said to be an asymptotic fixed point of T if C contain a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$ and $\hat{F}(T)$ denote the set of asymptotic fixed points of T ;
(2) quasi-Bregman nonexpansive with respect to f [37], provided that

$$F(T) \neq \emptyset \;\; and \;\; D_f(p, Tx) \leq D_f(p, x), \; \forall x \in C, p \in F(T).$$

(3) Bregman relatively nonexpansive with respect to f [37], provided that

$$\hat{F}(T) = F(T) \;\; and \;\; D_f(p, Tx) \leq D_f(p, x), \; \forall x \in C, p \in F(T).$$

(4) Bregman strongly nonexpansive with respect to f and $\hat{F}(T)$ provided that

$$F(T) \neq \emptyset \;\; and \;\; D_f(p, Tx) \leq D_f(p, x), \; \forall x \in C, p \in \hat{F}(T)$$

and, if whenever $\{x_n\} \subset C$ is bounded, $p \in \hat{F}(T)$ and

$$\lim_{n\to 0}(D_f(p, x_n) - D_f(p, Tx_n)) = 0$$

Consequently

$$\lim_{n\to 0} D_f(x_n, Tx_n) = 0.$$

(5) Bregman quasi-nonexpansive, provided that $F(T) \neq \emptyset$ and

$$D_f(p, Tx) \leq D_f(p, x), \;\; \forall x \in C, p \in F(T).$$

(6) Bregman quasi- asymptotically nonexpansive [37], provided that $F(T) \neq \emptyset$ and there exists a real number $\{k_n\} \subset [1, \infty), k_n \longrightarrow 1$ as $n \to \infty$ such that

$$D_f(p, T^n x) \leq k_n D_f(p, x), \quad \forall n \geq 1, x \in C, p \in F(T).$$

(7) Closed, if for any sequence $\{x_n\} \subset C$ with $x_n \longrightarrow x \in C$ and $Tx_n \longrightarrow y$ $(y \in C)$, then $y = Tx$.

Definition 1.4. [37], (1) A countable family of mappings $\{T_i\} : E \longrightarrow E$ is said to be uniformly Bregman quasi- asymptotically nonexpansive, if $\Gamma = \cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and there exists a real number $\{k_n\} \subset [1, \infty), k_n \longrightarrow 1$ as $n \to \infty$ such that

$$D_f(p, T_i^n x) \leq k_n D_f(p, x), \quad \forall n \geq 1, i \geq 1 x \in E, p \in \Gamma.$$

(2) A mappings $T : E \longrightarrow E$ is said to be uniformly L- Lipschits continuous, if there exists a constant $L > 0$ such that

$$\| T^n x - T^n y \| \leq L \| x - y \|, \ \forall x, y \in E, \forall n \geq 1.$$

In 2013, Zhao et al. [37], developed the following algorithm for approximating a common fixed point of Bregman quasi asymptotically nonexpansive mappings. They applied their results for the solution of generalised mixed equilibrium problem in reflexive Banach spaces:

$$\begin{cases} x_1 \in E \ chosen \ arbitrary : C_1 = C, , \\ y_{n,m} = \nabla f^* \big[\beta_n \nabla f(x_1) + (1 - \beta_n) \nabla f(S_m^n x_n) \big], \ m \geq 1, \\ C_{n+1} = \{ v \in C_n : \sup\limits_{m \geq 1} D_f(v, y_{n,m}) \leq \beta_n D_f(v, x_1) + (1 - \beta_n) D_f(v, x_n) + \xi_n, \\ x_{n+1} = P_{C_{n+1}}^f (x_1), \forall n \geq 1, \end{cases}$$

where $\xi_n = (k_n - 1) \sup\limits_{m \geq 1} D_f(p, x_n)$, $P_{C_{n+1}}^f$ is the Bregman projection of E onto C_{n+1}. If $\Gamma = \cap_{i=1}^{\infty} F(T_i)$ is bounded, then $\{x_n\}$ converges strongly to $P_\Gamma^f(x_1)$.

In 2014, by considering shrinking projection technique, Tomizawa [33] established strong convergence theorem for Bregman quasi-asymptotically nonexpansive mappings in the intermediate sense in a reflexive Banach spaces. Darvish [20] studied strong convergence result for weak Bregman relatively nonexpansive mapping and the set of solutions of generalized mixed equilibrium problems in Banach spaces. Kazmi et al. [23] introduced hybrid iterative method to approximate a common solution for a system of a generalized equilibrium problem and a fixed point problem for a Bregman relatively nonexpansive mapping in reflexive Banach spaces.
Recently, Alansari et al [1], consider the following inertial algorithm for solving a common solution for a system of variational inequality problem, generalized equilibrium problem and a common fixed point problem of relatively nonexpansive mapping in Banach space. Let the sequences $\{x_n\}$, and $\{z_n\}$ be generated by the algorithm:

$$\begin{cases} x_0 = x_1, \ \ z_0 \in C, \ \ C_0 := C; \\ \mu_n = x_n + \sigma_n (x_n - x_{n-1}) \\ u_n = \Pi_C J^{-1} (J \mu_n - \omega_n D \mu_n); \\ y_n = J^{-1} (\beta_n J z_n + (1 - \beta_n) J S u_n); \\ z_{n+1} = S_{r_n} y_n, \\ M_n = \{ z \in C : \phi(z, z_{n+1}) \leq \beta_n \phi(z, z_n) + (1 - \beta_n) \phi(z, \mu_n) \}; \\ N_n = \{ z \in C : \langle x_n - z, J x_n - J x_0 \rangle \leq 0 \}; \\ x_{n+1} = \Pi_{M_n \cap N_n} x_0, \forall n \geq 0, \end{cases}$$

where $\{\beta_n\} \in [0,1],\ r_n \in [a,\infty)$ for some $a > 0,\ \{\sigma_n\} \in (0,1)$ and $\{\omega_n\} \in (0,\infty)$. Then sequences $\{x_n\}$, converges strongly to a point $\hat{x} \in \Gamma$.
Very recently, Jantakan and Kaewcharoen [21], established strong convergence result by approximating a common solution for a mixed equilibrium problem and common fixed point of a countable family of Bregman relatively nonexpansive mappings using the algorithm below:

$$\begin{cases} x_1 \in C, S_i x_1 = z_1^i \in C; \\ y_n^i = \nabla f^*(\gamma_n \nabla f(z_n^i) + (1-\gamma_n)\nabla f(S_i x_n)); \\ z_{n+1}^i = Res_{G,\psi}^f(y_n^i); \\ C_n^i = \{z \in C : \phi(z, z_{n+1}^i) \le \gamma_n \phi(z, z_n^i) + (1-\gamma_n)\phi(z, x_n)\}, \\ C_n = \cap_{i=1}^N C_n^i; \\ Q_n = \{z \in C : \langle \nabla f(x_1) - f(x_n), z - x_n \rangle \le 0\}, \\ x_{n+1} = Proj_{C_n \cap Q_n}^f x_1, \forall n \ge 1, \end{cases}$$

where $\{x_n\}$ is a sequence in $[0,1]$ such that $\lim\limits_{n\to\infty} \gamma_n = 0$. Then, the sequence $\{x_n\}$ converges strongly to $Proj_\Omega^f x_1$, where $Proj_\Omega^f x_1$ is the projection of C onto Ω

In this article, motivated by the results of Zhao et al. [37], Alansari et al [1] and Jantakan and Kaewcharoen [21] mentioned above, we consider the algorithm that combine $GMEP$ with the class of Bregman nonexpansive mappings and also incorporate inertial extrapolation technique for accelerating the convergence of the sequence of an algorithm. This lead to the investigation for approximate common solution of generalized mixed equilibrium problems and fixed points problem for Bregman quasi-asymptotically nonexpansive mappings in a reflexive Banach spaces. Our result modify and improves the result of Jantakan and Kaewcharoen [21] and many recently announced results.

2 Preliminaries

Let E be a reflexive Banach space and $f : E \longrightarrow (-\infty, +\infty]$ be a Gateaux differentiable and convex function. The Bregman projection [5] of $x \in intdomf$ onto a nonempty, closed and convex set $C \subset domf$ is the necessarily unique vector $proj_C^f(x)$ satisfying

$$D_f(proj_C^f(x), x) := \inf\{D_f(y,x) : y \in C\}.$$

The modulus of total convexity of f at $x \in int\ domf$ is the function $v_f(x,t) : [0,+\infty) \longrightarrow [0,+\infty)$ defined by

$$v_f(x,t) := \inf\{Df(y,x) : y \in domf, \|y - x\| = t\}.$$

The function f is called totally convex at x, if $v_f(x,t) > 0$ whenever $t > 0$. The function f is called totally convex, if it is totally convex at any point $x \in int\ domf$ and it is said to be totally convex on bounded sets, if $v_f(B,t) > 0$, for any nonempty bounded subset B of E and $t > 0$, where the modulus of the total convexity of the function f on the set B is the function $v_f : int\ domf \times [0,+\infty) \longrightarrow [0,+\infty)$ defined by

$$v_f(B,t) := \inf\{v_f(x,t) : x \in B \cap domf\}.$$

Recall that f is totally convex on bounded sets if and only if f is uniformly convex on bounded sets [10]

Lemma 2.1. *[10] Let C be a nonempty, closed and convex subset of a reflexive Banach space E. Let $f : E \longrightarrow (-\infty, +\infty]$ be a Gâteaux differentiable and totally convex function and let $x \in E$. Then:*
i) $z = P_C^f(x)$ if and only if $\langle \nabla f(x) - \nabla f(z), y - z \rangle \le 0,\ \forall y \in C$;
ii) $D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) \le D_f(y,x),\ \forall y \in C$.

Let $f : E \longrightarrow (-\infty, +\infty]$ be a strong coercive Bregman function. Following [2] and [14], let the function $V_f : E \times E^* \longrightarrow [0, +\infty)$ asssociated with f is defined by

$$V_f(x, x^*) = f(x) + f^*(x^*) - \langle x, x^* \rangle, \forall x \in E,\ x^* \in E^*.$$

Then V_f is nonnegative and the following assertions hold:
(1) $V_f(x, x^*) = D_f(x, \nabla f^*(x^*))$ for all $x \in E$ and $y^* \in E^*$.
(2) $V_f(x, x^*) + \langle \nabla f^*(x^*) - x, y^* \rangle \leq V_f(x, x^* + y^*)$ for all $x \in E$ and $y^* \in E^*$.

Lemma 2.2. *[26] Let E be a Banach space and $f : E \longrightarrow (-\infty, +\infty]$ be a Gateaux differentiable) function which is uniformly convex on bounded subsets of E. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be bounded sequences in E. Then*

$$\lim_{t\to\infty} D_f(x_n, y_n) = 0\, if\ and\ only\ if \lim_{t\to\infty} \|x_n - y_n\| = 0.$$

Lemma 2.3. *[10] If $x \in dom\ f$, then the following statements are equivalent:*
i) the function f is totally convex at x;
ii) for any sequence $\{y_n\} \subset dom\ f$,

$$\lim_{t\to+\infty} D_f(y_n, x) = 0 \Longrightarrow \lim_{t\to+\infty} \parallel y_n - x \parallel = 0.$$

The function f is called sequentially consistent [10] if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that the first one is bounded

$$\lim_{t\to+\infty} D_f(y_n, x_n) = 0 \Longrightarrow \lim_{t\to+\infty} \parallel y_n - x_n \parallel = 0.$$

Lemma 2.4. *[8] Let $f : E \longrightarrow (-\infty, +\infty]$ be a convex function whose domain contains at least two points. Then, f is sequentially consistent if and only if it is totally convex on bounded sets.*

Lemma 2.5. *[29] Let $f : E \longrightarrow (-\infty, +\infty]$ be a Gateaux differentiable and totally convex function. If $x_1 \in E$ and the sequence $\{D_f(x_n, x_1)$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

Lemma 2.6. *[23] Let $f : E \longrightarrow (-\infty, +\infty]$ be a Legendre function and C be a nonempty, closed and convex subset of $\mathrm{int}\ dom f$. Let $T : C \longrightarrow C$ be a Bregman quasi asymptotically nonexpansive mapping with respect to f. Then $F(T)$ is closed and convex.*

Lemma 2.7. *[29] Let $f : E \longrightarrow (-\infty, +\infty]$ be a Gateaux differentiable and totally convex function, $x_1 \in E$ and C be a nonempty, closed and convex subset of E. Suppose that the sequence $\{x_n\}$ is bounded and any weak subsequential limit of $\{x_n\}$ belongs to C. If $D_f(x_n, x_1) \leq D_f(P_C^f x_1, x_1)$ for any $n \in N$, then $\{x_n\}$ strongly converges to $P_C^f x_1$.*

Lemma 2.8. *[9] Let $f : E \longrightarrow (-\infty, +\infty]$ be a Legendre function. Then, the function f is totally convex on bounded sets if and only if f is uniformly convex on bounded subsets of E.*

Lemma 2.9. *[35] Let $f : E \longrightarrow \mathbb{R}$ be a strong coercive and uniformly convex on bounded subsets of E, then f^* is bounded and uniformly Frechet differentiable on bounded subsets of E^*.*

Lemma 2.10. *[31] Let $f : E \longrightarrow (-\infty, +\infty]$ be a proper convex and lower semi-continuous Legendre function, then $f^* : E^* \longrightarrow (-\infty, +\infty]$ is a proper $weak^*$ lower semi continuous and convex function. Thus for any $z \in E$, for any $\{x_n\} \subset E$ and $\{t_i\}_{i=1}^N \subset (0, 1)$ with $\sum_{i=1}^N t_i = 1$, the following holds*

$$D_f\big(z, \nabla f^*\big(\sum_{i=1}^N t_i \nabla f(x_i)\big)\big) \leq \sum_{i=1}^N t_i D_f(z, x_i).$$

Lemma 2.11. *[28] Let $f : E \longrightarrow (-\infty, +\infty]$ be uniformly Frechet differentiable and bounded on bounded subsets of E. Then, f is uniformly continuous on bounded subsets of E and ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

Consider $\Theta : C \times C \longrightarrow \mathbb{R}$ as a bifunction satisfies the following assumptions [12]::
(A_1) $\Theta(x, x) = 0, \forall x \in C$;
(A_2) Θ is monotone, 1.e, $\Theta(x, y) + \Theta(y, x) \leq 0,\ \forall x, y \in C$;
(A_3) For each $x, y, z \in C, \limsup_{t\to 0} \Theta(tz + (1-t)x, y) \leq \Theta(x, y)$;
(A_4) For each $x \in C, y \mapsto \Theta(x, y)$ is convex and lower semicontinuous.

Lemma 2.12. *[21] Let C be a nonempty, closed and convex subset of a real reflexive Banach space E. Let $\Theta : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying $(A_1) - (A_4)$, $\psi : C \longrightarrow E^*$ be a continuous monotone mapping and $\varphi : C \longrightarrow \mathbb{R} \cup \{\infty\}$ be proper convex and lower semi-continuous. Let $f : E \longrightarrow (-\infty, +\infty]$ be a coercive Legendre function. For $x \in E$ and define a mapping $Res^f_{\Theta,\psi,\varphi} : E \longrightarrow 2^C$ as follws:*

$$\begin{aligned} Res^f_{\Theta,\psi,\varphi}(x) &= \{z \in C : \Theta(z, y) + \langle \psi z, y - z\rangle + \varphi(y) - \varphi(z) \\ &+ \langle \nabla f(z) - \nabla f(x), y - z\rangle \geq 0,\ \forall y \in C\} \end{aligned}$$

Then, the following properties holds:
(a) $Res^f_{\Theta,\psi,\varphi}$ is single-valued and $dom\ (Res^f_{\Theta,\psi,\varphi}) = E$;
(b) $Res^f_{\Theta,\psi,\varphi}$ is Bregman firmly nonexpansive type mapping;
(c) $F(Res^f_{\Theta,\psi,\varphi}) = GMEP(\Theta, \psi, \varphi)$;
(d) $GMEP(\Theta, \psi, \varphi)$ is closed and convex;
(e) $D_f(p, Res^f_{\Theta,\psi,\varphi}(x)) + D_f(Res^f_{\Theta,\psi,\varphi}(x), x) \leq D_f(p, x),\ \ \forall p \in F(Res^f_{\Theta,\psi,\varphi}),\ \ x \in E.$
It is well known that $Res^f_{\Theta,\psi,\varphi}$ is a closed Bregman quasi-nonexpansive mapping, and so is a closed Bregman quasi-asymptotically nonexpansive mapping.

3 Strong convergence theorem

Theorem 3.1. *Let E be a reflexive Banach space with dual E^* and C be a nonempty closed and convex subset of E such that $C \subset int\ (dom f)$. Let $f : E \longrightarrow (-\infty, +\infty]$ be a strong coercive legendre function which is bounded, uniformly Frechet differentiable and totally convex on bounded subset of E. Let $\Theta : C \times C \longrightarrow \mathbb{R}$ be bi functions which satisfying assumptions $(A_1) - (A_4)$, $\psi : C \longrightarrow E^*$ be continuous monotone mappings and $\varphi : C \longrightarrow \mathbb{R}$ be convex and lower semi-continuous functions. Let $\{T_j\}_{j=1}^N : E \longrightarrow E$ be a countable family of closed and uniformly Bregman quas- asymptotically nonexpansive mappings and for each $j \geq 1, T_j$ is uniformly L_j- Lipschitz continuous. Assume that $\Omega := \left(\cap_{j=1}^N F(T_j)\right) \cap \left(GMEP(\Theta, \psi, \varphi)\right) \neq \emptyset$. Let $\{x_n\}$ and $\{z_n\}$ be sequences generated by the iterative schemes:*

$$\begin{cases} x_1, z_1^j \in C = E \\ w_n = \nabla f^*(\nabla f(x_n) + \theta_n \nabla f(x_n - x_{n-1})); \\ y_n^j = \nabla f^*(\beta_n \nabla f(T_j^n w_n) + (1-\beta_n)\nabla f(w_n)); \\ u_n^j = \nabla f^*(\alpha_n \nabla f(z_n^j) + (1-\alpha_n)\nabla f(y_n^j)); \\ z_{n+1}^j = Res^f_{\Theta,\psi,\varphi}(u_n^j); \\ C_n^j = \{z \in C : D_f(z, z_{n+1}^j) \leq \alpha_n D_f(z, z_n^j) + (1-\alpha_n) D_f(z, w_n) \\ +\beta_n \xi_n\}; \\ C_n = \cap_{j=1}^N C_n^j; \\ Q_n = \{z \in C : \langle \nabla f(x_1) - \nabla f(x_n), z - x_n\rangle \leq 0\}; \\ x_{n+1} = Proj^f_{C_n \cap Q_n} x_1,\ \ \forall n \geq 1, \end{cases} \tag{2}$$

where $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0, 1]$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\liminf_{n\to\infty}(1-\alpha_n)\beta_n > 0$. $\theta_n(x_n - x_{n-1})$ is the inertial term with $\theta_n \in (0, 1)$ and $\xi_n = (k_n - 1)D_f(q, w_n)$. Then, $\{x_n\}$ converges strongly to

$Proj^f_\Omega x_1$, *where* $Proj^f_\Omega x_1$ *is the Bregman projection of* C *onto* Ω.

Proof.: Let two functions $\Gamma : C \times C \longrightarrow \mathbb{R}$ and $Res^f_{\Theta,\psi,\varphi} : E \longrightarrow 2^C$ be defined by

$$\Gamma(z, y) = \Theta(z, y) + \langle \psi z, y - z\rangle + \varphi(y) - \varphi(z), \forall y \in C$$

and

$$Res^f_{\Theta,\psi,\varphi}(x) = \{z \in C : \Gamma(z, y) + \langle \nabla(z) - \nabla f(x), y - z\rangle \geq 0, \forall y \in C\}, \forall x \in E$$

respectively. Consequently by Lemma 2.12 that, the function Γ satisfies assumptions $(A_1) - (A_4)$ and $Res^f_{\Theta,\psi,\varphi}$ has the properties $(a) - (e)$.

We present the proof as follows:

Step 1 : We show that Ω is closed and convex. In view of Lemma 2.6, we get that $F(T_j)$ is closed and convex for all $j = 1, 2, ..., N$, this implies that $\cap_{j=1}^N F(T_j)$ is closed and convex and also by Lemma 2.12(c), we obtain that $GMEP(\Theta, \psi, \varphi)$ is closed and convex and therefore $\Omega := \cap_{j=1}^N F(T_j) \cap GMEP(\Theta, \psi, \varphi)$ is closed and convex.

Step 2 : We show that $C_n \cap Q_n$ is closed and convex for all n. Now, we first show that Q_n is convex for all $n \geq 1$. Assume that $a_1, a_2 \in Q_n$, $\lambda \in [0, 1]$ and $\omega = \lambda a_1 + (1 - \lambda)a_2$. Now

$$\langle \nabla f(x_1) - \nabla f(x_n), a_1 - x_n\rangle \leq 0, \tag{3}$$

and

$$\langle \nabla f(x_1) - \nabla f(x_n), a_2 - x_n\rangle \leq 0. \tag{4}$$

Multiplying λ and $(1 - \lambda)$ on both sides of (3) and (4) respectively, we conclude that

$$\langle \nabla f(x_1) - \nabla f(x_n), \lambda a_1 + (1 - \lambda)a_2 - x_n\rangle \leq 0,$$

by definition of ω, we have

$$\langle \nabla f(x_1) - \nabla f(x_n), \omega - x_n\rangle \leq 0.$$

Hence, $\omega \in Q_n$ and Q_n is convex. Let $\{\mu_m\}$ be a sequence in Q_n and $\mu_m \longrightarrow \mu$ as $m \to \infty$. By the definition of Q_n, we obtain

$$\langle \nabla f(x_1) - \nabla f(x_n), \mu_m - x_n\rangle \leq 0,$$

which gives that

$$\langle \nabla f(x_1) - \nabla f(x_n), \mu_m - \mu\rangle + \langle \nabla f(x_1) - \nabla f(x_n), \mu - x_n\rangle \leq 0.$$

By taking limit as $m \longrightarrow \infty$, we get

$$\langle \nabla f(x_1) - \nabla f(x_n), \mu - x_n\rangle \leq 0.$$

Therefore $\mu \in Q_n$, which implies that Q_n is closed for all $n \geq 1$. Also, we show that C_n is closed for all $n \geq 1$. Assume that $\{v_m\}$ is a sequence in C_n and $v_m \longrightarrow v$ as $m \longrightarrow \infty$. Then $\{v_m\}$ is in C^j_n for all $j = 1, 2, ..., N$, it follows from the definition of C^j_n in (2), that

$$\begin{aligned} D_f(v_m, z^j_{n+1}) \;\; &\leq \;\; \alpha_n D_f(v_m, Z^j_n) + (1 - \alpha_n)D_f(v_m, w_n) + \beta_n \xi_n, \\ &\qquad \forall n \geq 1,\; j = 1, 2, ...N. \end{aligned} \tag{5}$$

Now, by definition of the Bregman distance with respect to f, we present

$$\begin{aligned} f(v_m) - f(z^j_{n+1}) &- \langle \nabla f(z^j_{n+1}), v_m - z^j_{n+1}\rangle \leq \alpha_n\big(f(v_m) - f(z^j_{n+1}) - \langle \nabla f(z^j_n, v_m - z^j_n\rangle\big) \\ &+ (1-\alpha_n)\big(f(v_m) - f(w_n) - \langle \nabla f(w_n), v_m - w_n\rangle\big) + \beta_n\xi_n, \end{aligned} \tag{6}$$

also

$$\begin{aligned} \alpha_n\langle \nabla f(z^j_n), v_m - z^j_n\rangle &+ (1-\alpha_n)\langle \nabla f(w_n), v_m - w_n\rangle - \langle \nabla f(z^j_{n+1}), v_m - z^j_{n+1}\rangle \\ &\leq f(z^j_{n+1}) - \alpha_n f(z^j_n) - (1-\alpha_n)f(w_n) + \beta_n\xi_n. \end{aligned} \tag{7}$$

It implies that

$$\begin{aligned} \alpha_n\big[\langle f(z^j_n), v_m - v\rangle + \langle f(z^j_n), v - z^j_n\rangle\big] &+ (1-\alpha_n)\big[\langle f(w_n), v_m - v\rangle + \langle f(w_n), v - w_n\rangle\big] \\ &- \langle \nabla f(z^j_{n+1}), v_m - v\rangle - \langle \nabla f(z^j_{n+1}), v - z^j_{n+1}\rangle \\ &\leq f(z^j_{n+1}) - \alpha_n f(z^j_n) - (1-\alpha_n)f(w_n) + \beta_n\xi_n. \end{aligned}$$

Taking the advantage of the limit as $m \to \infty$, we arrive at

$$\begin{aligned} \alpha_n\langle \nabla f(z^j_n), v - z^j_n\rangle &+ (1-\alpha_n)\langle \nabla f(w_n), v - w_n\rangle - \langle \nabla f(z^j_{n+1}), v - z^j_{n+1}\rangle \\ &\leq f(z^j_{n+1}) - \alpha_n f(z^j_n) - (1-\alpha_n)f(w_n) + \beta_n\xi_n, \forall j = 1, 2, ..., N. \end{aligned}$$

Implies $v \in C^j_n$ for all $j = 1, 2, ..., N$. Hence, $v \in C_n$ and so C_n is closed. For any $a_1, a_2 \in C_n$, we obtain that $a_1, a_2 \in C^j_n$ for all $j = 1, 2, ..., N$. and $a_1, a_2 \in C$. Now since C is convex, $\omega = \lambda a_1 + (1-\lambda)a_2 \in C$ for $\lambda \in [0, 1]$. Taking the advantage of the definition of C_n, that

$$D_f(a_1, z^j_{n+1}) \leq \alpha_n D_f(a_1, z^j_n) + (1-\alpha_n)D_f(a_1, w_n) + \beta_n\xi_n, \tag{8}$$

and

$$D_f(a_2, z^j_{n+1}) \leq \alpha_n D_f(a_2, z^j_n) + (1-\alpha_n)D_f(a_2, w_n) + \beta_n\xi_n. \tag{9}$$

Using (5), (6) and (7), shows that (8) and (9) are equivalent to

$$\begin{aligned} \alpha_n\langle \nabla f(z^j_n), a_1 - z^j_n\rangle &+ (1-\alpha_n)\langle \nabla f(w_n), a_1 - w_n\rangle - \langle \nabla f(z^j_{n+1}), a_1 - z^j_{n+1}\rangle \\ &\leq f(z^j_{n+1}) - f(z^j_n) - (1-\alpha_n)f(w_n) + \beta_n\xi_n \end{aligned} \tag{10}$$

and

$$\begin{aligned} \alpha_n\langle \nabla f(z^j_n), a_2 - z^j_n\rangle &+ (1-\alpha_n)\langle \nabla f(w_n), a_2 - w_n\rangle - \langle \nabla f(z^j_{n+1}), a_2 - z^j_{n+1}\rangle \\ &\leq f(z^j_{n+1}) - f(z^j_n) - (1-\alpha_n)f(w_n) + \beta_n\xi_n \end{aligned} \tag{11}$$

By multiplying λ and $(1-\lambda)$ on both sides of (10) and (11) respectively, we get

$$\begin{aligned} \alpha_n\langle \nabla f(z^j_n), \lambda a_1 &+ (1-\lambda)a_2 - z^j_n\rangle + (1-\alpha_n)\langle \nabla f(w_n), \lambda a_1 + (1-\lambda)a_2 - w_n\rangle \\ &- \langle \nabla f(z^j_{n+1}), \lambda a_1 + (1-\lambda)a_2 - z^j_{n+1})\rangle \\ &\leq f(z^j_{n+1}) - f(z^j_n) - (1-\alpha_n)f(w_n) + \beta_n\xi_n, \forall j = 1, 2, ..., N. \end{aligned} \tag{12}$$

Considering the definition of ω and (12), we conclude that

$$\begin{aligned} \alpha_n\langle \nabla f(z^j_n), \omega - z^j_n\rangle &+ (1-\alpha_n)\langle \nabla f(w_n), \omega - w_n\rangle - \langle \nabla f(z^j_{n+1}), \omega - z^j_{n+1})\rangle \\ &\leq f(z^j_{n+1}) - f(z^j_n) - (1-\alpha_n)f(w_n) + \beta_n\xi_n, \forall j = 1, 2, ..., N, \end{aligned}$$

then $\omega \in C^j_n$ for all j= 1,2,...,N, therefore $\omega \in C_n$. Implies that C_n is closed and convex for all $n \geq 1$. Hence $C_n \cap Q_n$ is closed and convex for all $n \geq 1$.

Step 3 : We show that $\Omega \subset C_n \cap Q_n$ for all $n \geq 1$. Let $q \in \Omega$, from (2) and by Lemma 2.12, we have

$$\begin{aligned}
D_f(q, z_{n+1}^j) &= D_n\big(q, Res^f_{\Theta,\psi,\varphi}(u_n^j)\big) \\
&\leq D_f\big(q, u_n^j\big) - D_f\big(Res^f_{\Theta,\psi,\varphi}(u_n^j), u_n^j\big) \\
&\leq D_f\big(q, u_n^j\big) \\
&= D_f\big(q, \nabla f^*(\alpha_n \nabla(z_n^j) + (1-\alpha_n)\nabla f(y_n^j)\big) \\
&\leq \alpha_n D_f(q, z_n^j) + (1-\alpha_n) D_f(q, y_n^j) \\
&= \alpha_n D_f(q, z_n^j) + (1-\alpha_n)\big[D_f(q, \nabla f^*(\beta_n \nabla f(T_j^n w_n) + (1-\beta_n)\nabla f(w_n))\big] \\
&\leq \alpha_n D_f(q, z_n^j) + (1-\alpha_n)\big[\beta_n D_f(q, T_j^n w_n) + (1-\beta_n) D_f(q, w_n)\big] \\
&\leq \alpha_n D_f(q, z_n^j) + (1-\alpha_n)\big[\beta_n [k_n D_f(q, w_n)] + (1-\beta_n) D_f(q, w_n)\big] \\
&= \alpha_n D_f(q, z_n^j) + (1-\alpha_n)\big[\beta_n D_f(q, w_n) + \beta_n(k_n - 1) D_f(q, w_n) \\
&+ (1-\beta_n) D_f(q, w_n)\big] \\
&= \alpha_n D_f(q, z_n^j) + (1-\alpha_n)\big[D_f(q, w_n) + \beta_n \xi_n\big] \\
&= \alpha_n D_f(q, z_n^j) + (1-\alpha_n) D_f(q, w_n) + \beta_n \xi_n - \alpha_n \beta_n \xi_n \\
&\leq \alpha_n D_f(q, z_n^j) + (1-\alpha_n) D_f(q, w_n) + \beta_n \xi_n, \qquad (13)
\end{aligned}$$

where $\xi_n = (k_n - 1) D_f(q, w_n)$. So we have that $q \in C_n^j$ for all $j = 1, 2, .., N$ and then $q \in C_n = \cap_{j=1}^N C_n^j$. Hence $\Omega \subset C_n, \forall n \geq 1$. Next, we prove by induction that $\Omega \subset C_n \cap Q_n, \forall n \geq 1$. It Taking the advantage of the definition of Q_n, that $Q_1 = C$ we arrive at $\Omega \subset C_1 \cap Q_1$. Assume that $\Omega \subset C_k \cap Q_k$ for some $k > 0$. Since $C_k \cap Q_k$ is closed and convex, then there exists $x_{k+1} \in C_k \cap Q_k$ such that $x_{n+1} = Proj^f_{C_k \cap Q_k}(x_1)$. By using the definition of x_{k+1}, we obtain for all $z \in C_k \cap Q_k$ that

$$\langle \nabla f(x_1) - \nabla f(x_{k+1}), x_{k+1} - z \rangle \geq 0$$

Now, for $\Omega \subset C_k \cap Q_k$, we get that

$$\langle \nabla f(x_1) - \nabla f(x_{k+1}), x_{k+1} - q \rangle \geq 0, \forall q \in \Omega,$$

then $q \in Q_{k+1}$. Hence, we obtain $\Omega \subset C_{k+1} \cap Q_{k+1}$, now since $\Omega \subset C_n, \forall n \geq 1$. Thus, we conclude that $\Omega \subset C_n \cap Q_n, \forall n \geq 1$ and therefore $x_{n+1} = Proj^j_{C_n \cap Q_n}(x_1)$ is well defined $\forall n \geq 1$. This shows that $\{x_n\}$ is well defined

Step 4 : We show that the sequences $\{x_n\}$, $\{w_n\}$, $\{y_n^j\}_{j=1}^\infty$, $\{u_n^j\}_{j=1}^\infty$, $\{z_n^j\}_{j=1}^\infty$, and $\{T_j^n w_n\}_{j=1}^\infty, j = 1, 2, ...N$ are bounded. Now, by using the definition of Q_n and it follows from Lemma 2.1 that $x_n = Proj^f_{Q_n}(x_1)$, we have

$$\begin{aligned}
D_f(x_n, x_1) &= D_f(Proj^f_{Q_n}(x_1), x_1) \\
&\leq D_f(q, x_1) - D_f(q, Proj^f_{Q_n}(x_1)) \\
&\leq D_f(q, x_1), \forall q \in \Omega \subset Q_n.
\end{aligned}$$

Therefore $\{D_f(x_n, x_1)\}$ is bounded. Hence by Lemma 2.5, $\{x_n\}$ is bounded. Also, the inequality

$$\begin{aligned}
D_f(q, x_n) &= D_f(q, Proj^f_{C_{n-1} \cap Q_{n-1}}(x_1)) \\
&\leq D_f(q, x_1) - D_f(x_n, x_1) \\
&\leq D_f(q, x_1)
\end{aligned}$$

implies that $\{D_f(q, x_n)\}$ is bounded. Now by using the fact that $D_f(q, T_j^n x_n) \leq D_f(q, x_n), \forall q \in \Omega, j = 1, 2, ..., N$ then $\{D_f(q, T_j^n x_n)\}_{n=1}^\infty$ is bounded, for all $j = 1, 2, ...N$. Also in view of the that $D_f(q, T_j^n w_n) \leq k_n D_f(q, w_n), \forall q \in \Omega, j = 1, 2, ..., N$ therefore $\{w_n\}$, $\{y_n\}_{n=1}^\infty$, $\{u_n\}_{n=1}^\infty$, and

$\{T_j^n w_n\}_{n=1}^{\infty}$, are bounded. Now setting $M = max\{D_f(q, z_1^j), \ \sup_n D_f(q, w_n)\}$, then obviously $D_f(q, z_1^j) \leq M$. Let $D_f(q, z_n^j) \leq M$. for some n and $j = 1, 2, ..., N$ then by (13), we have

$$\begin{aligned} D_f(q, z_{n+1}^j) &\leq \alpha_n M + (1-\alpha_n)M + \beta_n \xi_n \\ &\leq M + \beta_n \xi_n \end{aligned}$$

Thus, $\{D_f(q, z_{n+1}^j)\}_{j=1}^{\infty}$ is bounded, and so $\{D_f(q, z_n^j)\}_{j=1}^{\infty}$. Also by Lemma 2.5 $\{z_n^j\}_{j=1}^{\infty}$ is also bounded for all $j = 1, 2, ..., N$.

Step 5 : We show that the sequences $\{x_n\}$, converges strongly to some point $\hat{x}$. Now since $x_{n+1} = Proj^f_{C_n \cap Q_n}(x_1)$ and $x_n = Proj^f_{Q_n}(x_1)$, we get that

$$D_f(x_n, x_1) \leq D_f(x_{n+1}, x_1), \forall n \geq 1.$$

This implies that $\{D_f(x_n, x_1)\}$ is non decreasing. Since $\{D_f(x_n, x_1)\}$ is bounded, then $\lim_{n\to\infty} D_f(x_n, x_1)$ exists. On the other hand, we obtain

$$\begin{aligned} D_f(x_{n+1}, x_n) &= D_f(x_{n+1}, Proj^f_{Q_n}(x_1)) \\ &\leq D_f(x_{n+1}, x_1) - D_f(Proj^f_{Q_n}(x_1), x_1) \\ &= D_f(x_{n+1}, x_1) - D_f(x_n, x_1). \end{aligned}$$

Therefore

$$\lim_{n\to\infty} D_f(x_{n+1}, x_n). \tag{14}$$

Since f is totally convex on bounded sets, f is sequentially consistent, mow by Lemma 2.3 and (14), we conclude that

$$\lim_{n\to\infty} \| x_{n+1} - x_n \| = 0. \tag{15}$$

Now, since $\{x_n\}$ is bounded and E is reflexive, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x}$ as $k \to \infty$(some point in C). Since C_n and Q_n are closed and convex, then $C_n \cap Q_n$ is closed and convex, implies that C_n is weakly closed and $\hat{x} \in C_n$ for each $n \geq 1$. It follows from $x_{n_k} = Proj^f_{Q_{n_k}}(x_1)$ that

$$D_f(x_{n_k}, x_1) \leq D_f(\hat{x}, x_1), \forall n_k \geq 1.$$

Since f is a lower semi-continuous function on convex set C, then it implies weakly lower semi-continuous on C. Therefore, we have

$$\begin{aligned} \liminf_{n_k\to\infty} D_f(x_{n_k}, x_1) &= \liminf_{n_k\to\infty} \{f(x_{n_k}) - f(x_1) - \langle \nabla f(x_1), x_{n_k} - x_1 \rangle\} \\ &\geq f(\hat{x}) - f(x_1) - \langle \nabla f(x_1), \hat{x} - x_1 \rangle \\ &= D_f(\hat{x}, x_1). \end{aligned}$$

Also

$$\begin{aligned} D_f(\hat{x}, x_1) &\leq \liminf_{n_k\to\infty} D_f(x_{n_k}, x_1) \\ &\leq \limsup_{n_k\to\infty} D_f(x_{n_k}, x_1) \\ &\leq D_f(\hat{x}, x_1). \end{aligned}$$

It follows that

$$\lim_{n_k\to\infty} D_f(x_{n_k}, x_1) = D_f(\hat{x}, x_1)$$

and

$$f(x_{n_k}) \longrightarrow f(\hat{x}).$$

Since f is uniformly continuous, we conclude that

$$\lim_{n_k\to\infty} x_{n_k} = \hat{x}.$$

Also, since $\{D_f(x_n, x_1)\}$ is convergent and $\lim_{n_k\to\infty} D_f(x_{n_k}, x_1) = D_f(\hat{x}, x_1)$ implies that $\lim_{n\to\infty} D_f(x_n, x_1) = D_f(\hat{x}, x_1)$. If there exists another subsequence $\{x_{n_m}\} \subset \{x_n\}$ such that $x_{n_m} \longrightarrow x^*$, now by Lemma 2.1, we obtain

$$\begin{aligned} D_f(\hat{x}, x^*) &= \lim_{n_k,n_m\to\infty} D_f(x_{n_k}, x_{n_m}) \\ &= \lim_{n_k,n_m\to\infty} D_f\big(x_{n_k}, Proj^f_{Q_{n_m}}(x_1)\big) \\ &\leq \lim_{n_k,n_m\to\infty} \big(D_f(x_{n_k}, x_1) - D_f(Proj^f_{Q_{n_m}}(x_1), x_1)\big) \\ &= \lim_{n_k,n_m\to\infty} \big(D_f(x_{n_k}, x_1) - D_f(x_{n_m}, x_1)\big) \\ &= D_f(\hat{x}, x_1) - D_f(x^*, x_1). \end{aligned}$$

This implies that $\hat{x} = x^*$, therefore

$$\lim_{n\to\infty} x_n = \hat{x} \tag{16}$$

Step 6 : We show that $\hat{x} \in \Omega$, firstly we show that $\hat{x} \in \cap_{j=1}^N F(T_j)$. Now from the definition ξ_n, we have

$$\lim_{n\to\infty} \xi_n = \lim_{n\to\infty} \big((k_n - 1)D_f(q, w_n)\big) = 0.$$

Also, by definition of w_n, we get that

$$\nabla f(w_n) - \nabla f(x_n) = \theta_n(\nabla f(x_n) - \nabla f(x_{n-1}).$$

Which implies that

$$\begin{aligned} \lim_{n\to\infty} \parallel \nabla f(w_n) - \nabla f(x_n) \parallel &= \lim_{n\to\infty} \parallel \theta_n \nabla f(x_n - x_{n-1}) \parallel \\ &\leq \parallel \nabla f(x_n - x_{n-1}) \parallel \longrightarrow 0 \ \ as \ \ n \longrightarrow \infty. \end{aligned}$$

Thus

$$\lim_{n\to\infty} \parallel \nabla f(w_n) - \nabla f(x_n) \parallel = 0.$$

Since f is the legendre function and f^* is uniformly Frechet defferentiable on bounded subsets, we have

$$\lim_{n\to\infty} \parallel w_n - x_n \parallel = 0. \tag{17}$$

Since $\{w_n\}$ is bounded, by Lemma 2.2, we obtain

$$\lim_{n\to\infty} D_f(w_n, x_n) = 0.$$

Using (15) and (17), we get

$$\lim_{n\to\infty} \parallel x_{n+1} - w_n \parallel = 0. \tag{18}$$

It follows from Lemma 2.2, that

$$\lim_{n\to\infty} D_f(x_{n+1}, w_n) = 0. \tag{19}$$

By the three point identity of the Bregman distance, we have

$$D_f(x_{n+1}, z_n^j) = \langle \nabla f(z_n^j) - \nabla f(x_{n+1}), q - x_{n+1}\rangle + D_f(q, z_n^j) - D_f(q, x_{n+1}).$$

Since f is bounded on bounded subset of E, then ∇f is bounded on bounded subset of E^* and hence by the boundedness of $\{x_n\}$, $\{T_j^n w_n\}_{n=1}^{\infty}$, and $\{z_n\}_{n=1}^{\infty}$, the sequences $\{\nabla f(x_n)\}$, $\{\nabla f(T_j^n w_n)\}_{n=1}^{\infty}$, and $\{\nabla f(z_n)\}_{n=1}^{\infty}$ are bounded in E^*, which implies that $\left\{D_f(x_{n+1}, z_n^j)\right\}_{n=1}^{\infty}$ is bounded. Now since $x_{n+1} = Proj^f_{C_n\cap Q_n}(x_1) \in C_n$, we get

$$\begin{aligned} D_f(x_{n+1}, z_{n+1}^j) \leq \alpha_n D_f(x_{n+1}, z_n^j) + (1-\alpha_n) D_f(x_{n+1}, w_n) \\ +\beta_n \xi_n, \forall j = 1, 2, ..., N. \end{aligned} \tag{20}$$

Using (19), $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} \xi_n = 0$ in (20), we have

$$\lim_{n\to\infty} D_f(x_{n+1}, z_{n+1}^j) = 0, \forall j = 1, 2, ..., N.$$

Since f is totally convex on bounded sets, it follows from Lemma 2.3 that

$$\lim_{n\to\infty} \| x_{n+1} - z_{n+1}^j \| = 0, \forall j = 1, 2, ..., N. \tag{21}$$

By triangular inequality, we get

$$\| x_n - z_{n+1}^j \| \leq \| x_n - x_{n+1} \| + \| x_{n+1} - z_{n+1}^j \| .$$

Using (15) and (21), we obtain

$$\lim_{n\to\infty} \| x_n - z_{n+1}^j \| = 0. \tag{22}$$

Which implies that $z_{n+1}^j \longrightarrow \hat{x}$, since $x_n \longrightarrow \hat{x}$. Taking into account that

$$\| w_n - z_{n+1}^j \| \leq \| w_n - x_n \| + \| x_n - z_{n+1}^j \| .$$

Also, by (17) and (22), we have

$$\lim_{n\to\infty} \| w_n - z_{n+1}^j \| = 0 \tag{23}$$

Now, by Lemma 2.11, f and ∇f are uniformly continuous since f is uniformly Frechet differentiable on bounded subsets. Then using (23),we have

$$\lim_{n\to\infty} |f(w_n) - f(z_{n+1}^j)| = 0, \forall j = 1, 2, ..., N. \tag{24}$$

and

$$\lim_{n\to\infty} \|\nabla f(w_n) - \nabla f(z_{n+1}^j)\| = 0, \forall j = 1, 2, ..., N. \tag{25}$$

By definition of Bregman distance, we have

$$\begin{aligned} D_f(q, w_n) &- D_f(q, z_{n+1}^j) = f(q) - f(w_n) - \langle \nabla f(w_n), q - w_n\rangle \\ &- \left(f(q) - f(z_{n+1}^j) - \langle \nabla f(z_{n+1}^j), q - z_{n+1}^j\rangle\right) \\ &= f(z_{n+1}^j) - f(w_n) + \langle \nabla f(z_{n+1}^j), q - w_n\rangle \\ &+ \langle \nabla f(z_{n+1}^j), w_n - z_{n+1}^j\rangle - \langle \nabla f(w_n), q - w_n\rangle \\ &= f(z_{n+1}^j) - f(w_n) + \langle \nabla f(z_{n+1}^j) - \nabla f(w_n), q - w_n\rangle \\ &+ \langle \nabla (z_{n+1}^j), w_n - z_{n+1}^j\rangle \end{aligned} \tag{26}$$

Since $\{z_{n+1}^j\}_{n=1}^{\infty}$ and $\{\nabla f(z_{n+1}^j)\}_{n=1}^{\infty}$ are bounded for all $j = 1, 2, ...N$, using (23), (24), (25) in(26), we obtain

$$\lim_{n\to\infty}\left(D_f(q, w_n) - D_f(q, z_{n+1}^j)\right) = 0, \quad \forall j = 1, 2, ..., N. \tag{27}$$

Furthermore, by Lemma 2.12(e) and Lemma 2.10, for each $j = 1, 2, ...N$, we get

$$\begin{aligned}
D_f(z_{n+1}^j, u_n^j) &\leq D_f(q, u_n^j) - D_f(q, z_{n+1}^j)\\
&= D_f\big(q, \nabla f^*(\alpha_n \nabla f(z_n^j) + (1-\alpha_n)\nabla f(y_n^j)\big) - D_f(q, z_{n+1}^j)\\
&\leq \alpha_n D_f(q, z_n^j) + (1-\alpha_n) D_f\big(q, \nabla f^*(\beta_n \nabla f(T_j^n w_n) + (1-\beta_n)\nabla f(w_n)\big)\\
&- D_f(q, z_{n+1}^j)\\
&\leq \alpha_n D_f(q, z_n^j) + (1-\alpha_n)\big[\beta_n D_f(q, T_j^n w_n) + (1-\beta_n) D_f(q, w_n)\big]\\
&- D_f(q, z_{n+1}^j)\\
&\leq \alpha_n D_f(q, z_n^j) + (1-\alpha_n)\big[\beta_n (k_n D_f(q, w_n)) + (1-\beta_n) D_f(q, w_n)\big]\\
&- D_f(q, z_{n+1}^j)\\
&\leq \alpha_n D_f(q, z_n^j) + (1-\alpha_n)\big[\beta_n (1 + (k_n - 1)) D_f(q, w_n) + (1-\beta_n) D_f(q, w_n)\big]\\
&- D_f(q, z_{n+1}^j)\\
&\leq \alpha_n D_f(q, z_n^j) + (1-\alpha_n)\big[D_f(q, w_n) + \beta_n (k_n - 1) D_f(q, w_n)\big]\\
&- D_f(q, z_{n+1}^j)\\
&= \alpha_n D_f(q, z_n^j) + (1-\alpha_n) D_f(q, w_n) + \beta_n \xi_n - \alpha_n \beta_n \xi_n - D_f(q, z_{n+1}^j)\\
&\leq \alpha_n D_f(q, z_n^j) + (1-\alpha_n) D_f(q, w_n) + \beta_n \xi_n - D_f(q, z_{n+1}^j)\\
&= \alpha_n D_f(q, z_n^j) + D_f(q, w_n) - \alpha_n D_f(q, w_n) + \beta_n \xi_n - D_f(q, z_{n+1}^j)\\
&= \alpha_n\big(D_f(q, z_n^j) - D_f(q, w_n)\big) + \big(D_f(q, w_n) - D_f(q, z_{n+1}^j)\big) + \beta_n \xi_n.
\end{aligned} \tag{28}$$

Since $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \xi_n = 0$ then using (27) in (28), we obtain

$$\lim_{n\to\infty} D_f(z_{n+1}^j, u_n^j) = 0, \forall j = 1, 2, ...N.$$

Since f is totally convex on bounded sets, it follows from Lemma 2.3 that

$$\lim_{n\to\infty} \| z_{n+1}^j - u_n^j \| = 0, \forall j = 1, 2, ...N. \tag{29}$$

By triangular inequality, we get that

$$\| x_n - u_n^j \| \leq \| x_n - z_{n+1}^j \| + \| z_{n+1}^j - u_n^j \|$$

Using (22) and (29), we obtain

$$\lim_{n\to\infty} \| x_n - u_n^j \| = 0, \forall j = 1, 2, ...N. \tag{30}$$

Which implies that $u_n^j \longrightarrow \hat{x} \in C$, since $x_n \longrightarrow \hat{x} \in C$. Now, taking into account that

$$\| w_n - u_n^j \| \leq \| w_n - x_n \| + \| x_n - u_n^j \|$$

By (17) and (30), we obtain

$$\lim_{n\to\infty} \| w_n - u_n^j \| = 0 \tag{31}$$

Since f is uniformly Frechet differentiable then by Lemma 2.11 ∇f is uniformly continuous on bounded sets. Using (29) and (31), we conclude that

$$\lim_{n\to\infty} \parallel \nabla f(z_{n+1}^j) - \nabla f(u_n^j) \parallel = 0, \forall j = 1, 2, ...N, \tag{32}$$

and

$$\lim_{n\to\infty} \parallel \nabla f(w_n) - \nabla f(u_n^j) \parallel = 0, \forall j = 1, 2, ...N. \tag{33}$$

On the other hand, for each $j = 1, 2, ...N$, we have

$$\begin{aligned}
\parallel \nabla f(w_n) - \nabla f(u_n^j) \parallel &= \parallel \nabla f(w_n) - \nabla f\big(\nabla f^*(\alpha_n \nabla f(z_n^j) + (1-\alpha_n)\nabla f(y_n^j))\big) \parallel \\
&= \parallel \nabla f(w_n) - \big(\alpha_n \nabla f(z_n^j) + (1-\alpha_n)\nabla f(y_n^j)\big) \parallel \\
&= \parallel (1-\alpha_n)\nabla f(w_n) + \alpha_n \nabla f(w_n) - \alpha_n \nabla f(z_n^j) - (1-\alpha_n)\nabla f(y_n^j) \parallel \\
&= \parallel \alpha_n\big(\nabla f(w_n) - \nabla f(z_n^j)\big) + (1-\alpha_n)\big(\nabla f(w_n) - \nabla f(y_n^j)\big) \parallel \\
&= \parallel \alpha_n\big(\nabla f(w_n) - \nabla f(z_n^j)\big) + (1-\alpha_n)\big[\nabla f(w_n) - \nabla f\big(\nabla f^*(\beta_n \nabla f(T_j^n w_n) \\
&+ (1-\beta_n)\nabla f(w_n))\big)\big] \parallel \\
&= \parallel \alpha_n\big(\nabla f(w_n) - \nabla f(z_n^j)\big) + (1-\alpha_n)\big[\nabla f(w_n) - \big(\beta_n \nabla f(T_j^n w_n) \\
&+ (1-\beta_n)\nabla f(w_n)\big)\big] \parallel \\
&= \parallel \alpha_n\big(\nabla f(w_n) - \nabla f(z_n^j)\big) + (1-\alpha_n)\big[\beta_n\big(\nabla f(w_n) - \nabla f(T_j^n w_n)\big)\big] \\
&\geq (1-\alpha_n)\beta_n \parallel \nabla f(w_n) - \nabla f(T_j^n w_n) \parallel - \alpha_n \parallel \nabla f(w_n) - \nabla f(z_n^j) \parallel
\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned}
(1-\alpha_n)\beta_n \parallel \nabla f(w_n) - \nabla f(T_j^n w_n) \parallel &\leq \alpha_n \parallel \nabla f(w_n) - \nabla f(z_n^j) \parallel \\
&+ \parallel \nabla f(w_n) - \nabla f(u_n^j) \parallel, \quad \forall j = 1, 2, ...N.
\end{aligned} \tag{34}$$

Since $\liminf_{n\to\infty}(1-\alpha_n)\beta_n > 0 and \lim_{n\to\infty} \alpha_n = 0$, then using (33) in (34), we obtain

$$\lim_{n\to\infty} \parallel \nabla f(w_n) - \nabla f(T_j^n w_n) \parallel = 0, \quad \forall j = 1, 2, ...N. \tag{35}$$

Again for $j = 1, 2, ...N$, we have

$$\begin{aligned}
\parallel \nabla f(w_n) - \nabla f(y_n^j) \parallel &= \parallel \nabla f(w_n) - \nabla f\big(\nabla f^*(\beta_n \nabla f(T_j^n w_n) + (1-\beta_n)\nabla f(w_n))\big) \parallel \\
&= \parallel \nabla f(w_n) - \big(\beta_n \nabla f(T_j^n w_n) + (1-\beta_n)\nabla f(w_n)\big) \parallel \\
&= \parallel \beta_n\big(\nabla f(w_n) - \nabla f(T_j^n w_n)\big) \parallel \\
&\leq \beta_n \parallel \nabla f(w_n) - \nabla f(T_j^n w_n) \parallel
\end{aligned} \tag{36}$$

Using (35) in (36),we obtain

$$\lim_{n\to\infty} \parallel \nabla f(w_n) - \nabla f(y_n^j) \parallel = 0, \quad \forall j = 1, 2, ...N.$$

Since f is the legendre function and f^* is uniformly Frenchet differentiable on bounded subsets, we have

$$\lim_{n\to\infty} \parallel w_n - y_n^j \parallel = 0, \quad \forall j = 1, 2, ...N. \tag{37}$$

It follows from(16) and (17), that

$$\lim_{n\to\infty} w_n = \hat{x} \tag{38}$$

Using (37) and (38), we have that $y_n^j \longrightarrow \hat{x}$ (as $n \to \infty$). Since ∇f is uniformly continuous, using (38) we have $\nabla f(w_n) \longrightarrow \nabla f(\hat{x})$. Therefore by (35), we obtain that

$$\lim_{n\to\infty} \nabla f(T_j^n w_n) = \nabla f(\hat{x}).$$

Since ∇f is uniformly continuous, we have

$$\lim_{n\to\infty} T_j^n w_n = \hat{x}, \quad \forall j = 1, 2, ...N. \tag{39}$$

It follows from (18,) (29) and (37) that the sequence $\{x_n\}$, $\{y_n^j\}_{n=1}^{\infty}$, $\{u_n^j\}_{n=1}^{\infty}$, $\{w_n\}$, and $\{z_n^j\}_{n=1}^{\infty}$, for all $j = 1, 2, ...N$, have the some asymptotic behaviour, therefore by (39), we conclude that

$$\lim_{n\to\infty} T_j^n x_n = \hat{x}, \quad \forall j = 1, 2, ...N. \tag{40}$$

Now, by the assumptions that for all $j = 1, 2, ...N$, T_j is uniformly L_j- Lipschitz continuous, we get that

$$\begin{aligned} \| T_j^{n+1} x_n - T_j^n x_n \| &\leq \| T_j^{n+1} x_n - T_j^{n+1} x_{n+1} \| + \| T_j^{n+1} x_{n+1} - x_{n+1} \| \\ &+ \| x_{n+1} - x_n \| + \| x_n - T_j^n x_n \| \\ &\leq (L_j + 1) \| x_{n+1} - x_n \| + \| T_j^{n+1} x_{n+1} - x_{n+1} \| + \| x_n - T_j^n x_n \| \end{aligned} \tag{41}$$

By (40) and since $x_n \longrightarrow \hat{x}$ (as $n \longrightarrow \infty$), we obtain that

$$\lim_{n\to\infty} \| T_j^{n+1} x_n - T_j^n x_n \| = 0, \quad \forall j = 1, 2, ...N,$$

and

$$\lim_{n\to\infty} T_j^{n+1} x_n = \hat{x}, \quad \forall j = 1, 2, ...N,$$

which implies that

$$\lim_{n\to\infty} T_j T_j^n x_n = \hat{x}, \quad \forall j = 1, 2, ...N.$$

It follows from the closedness of T_j that

$$\lim_{n\to\infty} T_j \hat{x} = \hat{x}, \quad \forall j = 1, 2, ...N.$$

Therefore

$$\hat{x} \in \cap_{j=1}^{N} F(T_j).$$

Next, we show that $\hat{x} \in GMEP(\Theta, \psi, \varphi)$. Since $\{x_n\}$ is bounded and so is $\{\nabla f(x_n)\}$, there exists a subsequence $\{\nabla f(x_{n_k})\}$ of $\{\nabla f(x_n)\}$ such that $\nabla f(x_{n_k}) \rightharpoonup \hat{x} \in C$ as $k \to \infty$. It follows from (22) and (29) that there exists subsequence $\{\nabla f(z_{n_k}^j)\}$ of $\{\nabla f(z_n^j)\}$ and $\{\nabla f(u_{n_k}^j)\}$ of $\{\nabla f(u_n^j)\}$ such that $\nabla f(z_{n_k}^j) \rightharpoonup \hat{x} \in C$ and $\nabla f(u_{n_k}^j) \rightharpoonup \hat{x} \in C$ as $k \to \infty$, for all $j = 1, 2, ...N$ respectively. From $z_{n+1}^j = Res_{\Theta,\psi,\varphi}^f(u_n^j)$, we have

$$\Gamma(z_{n+1}^j, y) + \langle \nabla f(z_{n+1}^j) - \nabla f(u_n^j), y - z_{n+1}^j \rangle \geq 0, \forall y \in C.$$

Replacing n by n_k and using (A_2), we obtain

$$\langle \nabla f(z_{n_{k+1}}^j) - \nabla f(u_{n_k}^j), y - z_{n_{k+1}}^j \rangle \geq -\Gamma(z_{n_{k+1}}^j, y) \geq \Gamma(y, z_{n+1}^j), \forall y \in C, \quad \forall j = 1, 2, ...N.$$

Letting $k \longrightarrow \infty$, it follows from $\nabla f(z_{n_k}) \rightharpoonup \hat{x} \in C$ and (A_4) that

$$\Gamma(y, \hat{x}) \leq 0, \forall y \in C,$$

for λ with $0 < \lambda \leq 1$ and $y \in C$, assume that $y_\lambda = \lambda y + (1-\lambda)\hat{x}$. Since $y \in C$ and $\hat{x} \in C$, we have $y_\lambda \in C$ and $\Gamma(y_\lambda, \hat{x}) \leq 0, \forall y \in C$, so by (A_1) and (A_3), we obtain

$$\begin{aligned} 0 &= \Gamma(y_\lambda, y_\lambda) \\ &\leq \lambda\Gamma(y_\lambda, y) + (1-\lambda)\Gamma(y_\lambda, \hat{x}) \\ &\leq \lambda\Gamma(y_\lambda, y). \end{aligned}$$

Dividing by λ, we get

$$\Gamma(y_\lambda, y) \geq 0, \forall y \in C.$$

Letting $\lambda \longrightarrow 0$ and by (A_3), we conclude that

$$\Gamma(\hat{x}, y) \geq 0, \forall y \in C.$$

Thus

$$\hat{x} \in GMEP(\Theta, \psi, \varphi).$$

Therefore

$$\hat{x} \in \Omega := \big[\cap_{j=1}^{N} F(T_j)\big] \cap \big[GMEP(\Theta, \psi, \varphi)\big].$$

Step 7 : We show that $\hat{x} = Proj_\Omega^f(x_1)$. Since Ω is a nonempty closed and convex subset of E, $Proj_\Omega^f(x_1)$ is well defined. Suppose that $\hat{u} = Proj_\Omega^f(x_1)$, now by $x_{n+1} = Proj_{C_n \cap Q_n}^f(x_1)$ and $Proj_\Omega^f(x_1) \in \Omega \subseteq C_n \cap Q_n$, we conclude that

$$D_f(x_{n+1}, x_1) \leq D_f(\hat{u}, x_1)$$

Since $\{x_{n_k}\}$ is a weak convergence subsequence of $\{x_n\}$ and by Lemma 2.7, we get that the sequence $\{x_n\}$ converges strongly to $\hat{u}$. It follows from the uniqueness of the limit, we conclude that the sequence $\{x_n\}$ converges strongly to $\hat{x} = Proj_\Omega^f(x_1)$. This completes the proof.

Corollary 3.2. *Let E be a reflexive Banach space with dual E^* and C be a nonempty closed and convex subset of E such that $C \subset int\ (domf)$. Let $f : E \longrightarrow (-\infty, +\infty]$ be a strong coercive legendre function which is bounded, uniformly Frechet differentiable and totally convex on bounded subset of E. Let $\Theta : C \times C \longrightarrow \mathbb{R}$ be bi functions which satisfying assumptions $(A_1) - (A_4)$, $\psi : C \longrightarrow E^*$ be continuous monotone mappings. Let $\{T_j\}_{j=1}^N : E \longrightarrow E$ be a countable family of closed and uniformly Bregman quas- asymptotically nonexpansive mappings and for each $j \geq 1, T_j$ is uniformly L_j- Lipschitz continuous. Assume that $\Omega := \big(\cap_{j=1}^N F(T_j)\big) \cap \big(GMEP(\Theta, \psi)\big) \neq \emptyset$. Let $\{x_n\}$ and $\{z_n\}$ be sequences generated by the iterative schemes:*

$$\begin{cases} x_1, z_1^j \in C = E \\ w_n = \nabla f^*(\nabla f(x_n) + \theta_n \nabla f(x_n - x_{n-1})); \\ y_n^j = \nabla f^*(\beta_n \nabla f(T_j^n w_n) + (1-\beta_n)\nabla f(w_n)); \\ u_n^j = \nabla f^*(\alpha_n \nabla f(z_n^j) + (1-\alpha_n)\nabla f(y_n^j)); \\ z_{n+1}^j = Res_{\Theta,\psi}^f(u_n^j); \\ C_n^j = \{z \in C : D_f(z, z_{n+1}^j) \leq \alpha_n D_f(z, z_n^j) + (1-\alpha_n) D_f(z, w_n) \\ +\beta_n \xi_n\}; \\ C_n = \cap_{j=1}^N C_n^j; \\ Q_n = \{z \in C : \langle \nabla f(x_1) - \nabla f(x_n), z - x_n \rangle \leq 0\}; \\ x_{n+1} = Proj_{C_n \cap Q_n}^f x_1, \ \ \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$ *are sequences in* $[0,1]$ *such that* $\lim_{n\to\infty}\alpha_n = 0$ *and* $\liminf_{n\to\infty}(1-\alpha_n)\beta_n > 0$. $\theta_n(x_n - x_{n-1})$ *is the inertial term with* $\theta_n \in (0,1)$ *and* $\xi_n = (k_n - 1)D_f(q, w_n)$. *Then,* $\{x_n\}$ *converges strongly to* $Proj^f_{\Omega}x_1$.

Corollary 3.3. *Let* E *be a reflexive Banach space with dual* E^* *and* C *be a nonempty closed and convex subset of* E *such that* $C \subset int\,(domf)$. *Let* $f : E \longrightarrow (-\infty, +\infty]$ *be a strong coercive legendre function which is bounded, uniformly Frechet differentiable and totally convex on bounded subset of* E. *Let* $\Theta : C \times C \longrightarrow \mathbb{R}$ *be bi functions which satisfying assumptions* $(A_1) - (A_4)$. *Let* $\{T_j\}_{j=1}^N : E \longrightarrow E$ *be a countable family of closed and uniformly Bregman quas- asymptotically nonexpansive mappings and for each* $j \geq 1, T_j$ *is uniformly* L_j- *Lipschitz continuous. Assume that* $\Omega := \big(\cap_{j=1}^N F(T_j)\big) \cap \big(GMEP(\Theta)\big) \neq \emptyset$. *Let* $\{x_n\}$ *and* $\{z_n\}$ *be sequences generated by the iterative schemes:*

$$\begin{cases} x_1, z_1^j \in C = E \\ w_n = \nabla f^*(\nabla f(x_n) + \theta_n \nabla f(x_n - x_{n-1})); \\ y_n^j = \nabla f^*(\beta_n \nabla f(T_j^n w_n) + (1-\beta_n)\nabla f(w_n)); \\ u_n^j = \nabla f^*(\alpha_n \nabla f(z_n^j) + (1-\alpha_n)\nabla f(y_n^j)); \\ z_{n+1}^j = Res^f_{\Theta}(u_n^j); \\ C_n^j = \{z \in C : D_f(z, z_{n+1}^j) \leq \alpha_n D_f(z, z_n^j) + (1-\alpha_n)D_f(z, w_n) \\ +\beta_n \xi_n\}; \\ C_n = \cap_{j=1}^N C_n^j; \\ Q_n = \{z \in C : \langle \nabla f(x_1) - \nabla f(x_n), z - x_n\rangle \leq 0\}; \\ x_{n+1} = Proj^f_{C_n \cap Q_n} x_1, \ \ \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$ *are sequences in* $[0,1]$ *such that* $\lim_{n\to\infty}\alpha_n = 0$ *and* $\liminf_{n\to\infty}(1-\alpha_n)\beta_n > 0$. $\theta_n(x_n - x_{n-1})$ *is the inertial term with* $\theta_n \in (0,1)$ *and* $\xi_n = (k_n - 1)D_f(q, w_n)$. *Then,* $\{x_n\}$ *converges strongly to* $Proj^f_{\Omega}x_1$.

Corollary 3.4. *Let* E *be a reflexive Banach space with dual* E^* *and* C *be a nonempty closed and convex subset of* E *such that* $C \subset int\,(domf)$. *Let* $f : E \longrightarrow (-\infty, +\infty]$ *be a strong coercive legendre function which is bounded, uniformly Frechet differentiable and totally convex on bounded subset of* E. *Let* $\Theta : C \times C \longrightarrow \mathbb{R}$ *be bi functions which satisfying assumptions* $(A_1) - (A_4)$. *Let* $\{T_j\}_{j=1}^N : E \longrightarrow E$ *be a countable family of closed and uniformly Bregman relatively nonexpansive mappings and for each* $j \geq 1, T_j$ *is uniformly* L_j- *Lipschitz continuous. Assume that* $\Omega := \big(\cap_{j=1}^N F(T_j)\big) \cap \big(GMEP(\Theta)\big) \neq \emptyset$. *Let* $\{x_n\}$ *and* $\{z_n\}$ *be sequences generated by the iterative schemes:*

$$\begin{cases} x_1, z_1^j \in C = E \\ w_n = \nabla f^*(\nabla f(x_n) + \theta_n \nabla f(x_n - x_{n-1})); \\ y_n^j = \nabla f^*(\beta_n \nabla f(T_j^n w_n) + (1-\beta_n)\nabla f(w_n)); \\ u_n^j = \nabla f^*(\alpha_n \nabla f(z_n^j) + (1-\alpha_n)\nabla f(y_n^j)); \\ z_{n+1}^j = Res^f_{\Theta}(u_n^j); \\ C_n^j = \{z \in C : D_f(z, z_{n+1}^j) \leq \alpha_n D_f(z, z_n^j) + (1-\alpha_n)D_f(z, w_n); \\ C_n = \cap_{j=1}^N C_n^j; \\ Q_n = \{z \in C : \langle \nabla f(x_1) - \nabla f(x_n), z - x_n\rangle \leq 0\}; \\ x_{n+1} = Proj^f_{C_n \cap Q_n} x_1, \ \ \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$ *are sequences in* $[0,1]$ *such that* $\lim_{n\to\infty}\alpha_n = 0$ *and* $\liminf_{n\to\infty}(1-\alpha_n)\beta_n > 0$. $\theta_n(x_n - x_{n-1})$ *is the inertial term with* $\theta_n \in (0,1)$. *Then,* $\{x_n\}$ *converges strongly to* $Proj^f_{\Omega}x_1$.

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