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Numerical approximation of space fractional order wave equation using shifted Chebyshev polynomials of second and third kinds

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Abstract

In this paper, we proposed a numerical method to solve space fractional order wave equation (SFOWE). The method implements shifted Chebyshev polynomials of the second and third kinds with the fractional derivative interpreted in Caputo sense. Chebyshev collocation method is adopted to reduce the problem to a system of second order ordinary differential equation, while the finite difference method reduces that to a system of linear algebraic equations. The system of algebraic equations thus formed is solved with the aid of Mathematica 11.0. The proposed method gives results that compared favourably with existing results in the literature, while it is observed that the performance of second kind is relatively superior to that of the third kind polynomials. The results are presented in both tabular and 3D graphs, for ease of comparison.

1 Introduction

In recent years, a swift development on fractional calculus in theory and applications has become more apparent (1). Fractional calculus is a branch of mathematics that deals with generalization of differentiation and integration on non-integer orders (12). The history of fractional calculus dated back to the 17th century when mathematicians began exploring the concept of fractional order. However, the formal development of fractional calculus did not begin until 1790s, when French mathematician Joseph Liouville studied the properties of fractional order equations (14). In the mid-20th century, there was a renewed interest in fractional calculus with the advent of modern computers, as it was found to be useful in wide range of fields such as physics, engineering, and finance. Several researchers such as (12) and (11) advanced the theory of fractional calculus and its applications during this time. The fractional calculus has found its applications in the study of diffusion equations (9), wave propagation (2), signal processing (10) and other areas of mathematical physics. It has also been applied to the field of engineering, finance and biology (4). Seeking solution to fractional order differential equations remains a great task to be accomplished by the researchers, except in a very few number of these equations. Several attempts have been made by different researchers to solve the differential equation of non-integers values. Among the methods are, but not limited to Adomian Decomposition Method (ADM) (6), Homotopy Perturbation Method (HPM) (13), Collocation Method (7), Homotopy Analysis Method (HAM) (3; 15), Orthogonal Polynomial Approach (16), while (5) used Gengenbauer polynomial as an approach to the solution of fractional order partial differential equations. This paper focuses on the numerical solution of space fractional order wave equation by utilizing the accuracy of shifted Chebyshev polynomials of the second and the third kinds. Both polynomials in their shifted forms, along with the fractional derivative expressed in Caputo sense are used to transform the (SFOWE) into a system of second order ordinary differential equations while Finite Difference Method (FDM) reduces the system thus formed, to a system of linear algebraic equations which are later solved for the unknown constants.

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2 Preliminaries

2.1 Statement of the Problem

The class of problem considered in this paper is the space fractional order wave equation of the form:

$$\frac{\partial^2 \omega(x,t)}{\partial t^2} = \xi(x,t)\frac{\partial^\theta \omega(x,t)}{\partial x^\theta} + \eta(x,t), \tag{1}$$

on a finite domain $a < x < b$, $0 \leq t \leq T$. The parameter θ is the fractional order of partial differential equation (1) with $1 < \theta < 2$. The function $\eta(x,t)$ is a non-homogeneous source term and the coefficient function $\xi(x,t) \geq 0$. We also assumed the problem (1) is subject to the following initial conditions:

$$\omega(x,0) = \omega^0(x), \quad \omega_t(x,0) = \omega'(x) \tag{2}$$

and the Dirichlet boundary conditions

$$\omega(a,t) = \omega(b,t) = 0. \tag{3}$$

It is important to note that when $\theta = 2$, (1) becomes the classical wave equation of the form:

$$\frac{\partial^2 \omega(x,t)}{\partial t^2} = \xi(x,t)\frac{\partial^2 \omega(x,t)}{\partial x^2} + \eta(x,t) \tag{4}$$

2.2 Fractional Derivative

Let

$$f(x) = x^k \tag{5}$$

The first derivative is as usual

$$f^{'}(x) = \frac{d}{dx}f(x) = kx^{k-1} \tag{6}$$

Repeating this process gives more general result

$$\frac{d^\alpha}{dx^\alpha}x^k = \frac{k!}{(k-\alpha)!}x^{k-\alpha}. \tag{7}$$

Replacing the factorials by the gamma function equivalent, (7) becomes

$$\frac{d^\alpha}{dx^\alpha}x^k = \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}x^{k-\alpha}; \quad k > 0 \tag{8}$$

For $k = 2$ and $\alpha = \frac{1}{2}$, we obtain the half derivative of the function x^2 as

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}x^2 = \frac{\Gamma(2+1)}{\Gamma(2-\frac{1}{2}+1)}x^{2-\frac{1}{2}} = \frac{\Gamma(3)}{\Gamma(\frac{5}{2})}x^{\frac{3}{2}}.$$

2.3 The Caputo fractional derivative

The Caputo fractional derivative operator D^θ of order θ is defined in the following form:

$$D^\theta f(x) = \frac{1}{\Gamma(n-\theta)}\int_0^x (x-t)^{n-\theta-1}f^{(n)}(t)dt; \theta > 0 \tag{9}$$

where $n-1 < \theta < n$, $n \in N$, $x > 0$.
The linearity property of Caputo fractional derivative suit with that of the integer order differential operator.

$$D^\theta(\psi f(x) + \phi g(x)) = \psi D^\theta f(x) + \phi D^\theta g(x) \tag{10}$$

where ψ and ϕ are constants. For the Caputo derivative, it is possible to obtain the following results:

$$D^{\theta}x^{\tau} = \begin{cases} 0, & \text{for} \quad \tau \in N_0 \quad \text{and} \quad \tau < \lceil \theta \rceil \\ \frac{\Gamma(\tau+1)}{\Gamma(\tau+1-\theta)}x^{\tau-\theta}, & \text{for} \quad \tau \in N_0 \quad \text{and} \quad \tau \geq \lceil \theta \rceil, \end{cases} \tag{11}$$

where the function $\lceil \theta \rceil$ is the smallest integer greater than or equal to θ and N_0 represents the natural numbers with the inclusion of zero.

2.4 Proof:

We note that:

$$\begin{aligned} f^{(n)}(t^{\tau}) &= \tau f^{(n-1)}t^{\tau-1} \\ &= \tau(\tau-1)f^{(n-2)}t^{\tau-2} \\ &= \tau(\tau-1)(\tau-2)(t^{\tau-3})f^{(n-3)}t^{\tau-3} \\ \text{The process continues until} \\ &= \frac{\tau!}{(\tau-n)!}t^{\tau-n} \\ f^{(n)}(t^{\tau}) &= \frac{\Gamma(\tau+1)}{\Gamma(\tau+1-n)}t^{\tau-n} \end{aligned}$$

From (9) we have

$$\begin{aligned} D^{\theta}x^{\tau} &= \frac{1}{\Gamma(n-\theta)}\int_0^x (x-t)^{n-\theta-1}f^{(n)}(t^{\tau})dt \\ &= \frac{1}{\Gamma(n-\theta)}\int_0^x \frac{\Gamma(\tau+1)}{\Gamma(\tau+1-n)}t^{\tau-n}(x-t)^{n-\theta-1}dt \\ \text{Let } t = \mu x;\ 0 \leq \mu \leq 1,\ dt = xd\mu \\ &= \frac{1}{\Gamma(n-\theta)}\frac{\Gamma(\tau+1)}{\Gamma(\tau+1-n)}x^{\tau-\theta}\int_0^1 \mu^{\tau-n}(1-\mu)^{n-\theta-1}d\mu, \\ \text{But} \\ B(m,n) &= \int_0^x x^{m-1}(1-x)^{n-1}dx \\ &= \frac{1}{\Gamma(n-\theta)}\frac{\Gamma(\tau+1)}{\Gamma(\tau+1-n)}x^{\tau-\theta}B(\tau-n+1, n-\theta), \\ \text{Also note:} \\ B(m,n) &= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \\ &= \frac{1}{\Gamma(n-\theta)}\frac{\Gamma(\tau+1)}{\Gamma(\tau+1-n)}x^{\tau-\theta}\frac{\Gamma(\tau-n+1)\Gamma(n-\theta)}{\Gamma(\tau+1-\theta)} \\ \therefore D^{\theta}x^{\tau} &= \frac{\Gamma(\tau+1)}{\Gamma(\tau+1-\theta)}x^{\tau-\theta} \end{aligned}$$

3 Some Properties of Second and Third Kinds Chebyshev Polynomials

The Second and Third kinds Chebyshev polynomials denoted by $U_n(x)$ and $V_n(x)$ see [8] are orthogonal polynomials of degree n in x defined on $[-1, 1]$ and are of the form:
$U_n(x) = sin(n+1)\alpha/sin\alpha$ and $V_n(x) = cos(n+\frac{1}{2}\alpha)/cos\frac{1}{2}\alpha$ respectively, with $x = cos\alpha$ and

$\alpha \in [0, \pi]$.
The orthogonality of the second and third kinds polynomials are shown respectively as:

$$\langle U_n(x), U_m(x) \rangle = \int_{-1}^{1} \sqrt{1-x^2} \quad U_n(x)U_m(x)dx = \begin{cases} 0; & n \neq m, \\ \frac{\pi}{2}; & n = m, \end{cases} \tag{12}$$

$$\langle V_n(x), V_m(x) \rangle = \int_{-1}^{1} \sqrt{\frac{1+x}{1-x}} \quad V_n(x)V_m(x)dx = \begin{cases} 0; & n \neq m, \\ \pi; & n = m, \end{cases} \tag{13}$$

where $(1-x^2)^{\frac{1}{2}}$ and $(1+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$ are the weight functions respectively. The recurrence relations for these polynomials are given as:

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), \quad n = 2, 3, ... \tag{14}$$

with the first two terms given as $U_0(x) = 1$ and $U_1(x) = 2x$. Also,

$$V_n(x) = 2xV_n(x) - V_{n-1}(x), \quad n = 1, 2, ... \tag{15}$$

with the first two terms given as $V_0(x) = 1$ and $V_1(x) = 2x - 1$. The second kind shifted Chebyshev polynomial on the general intervals $[a, b]$ is obtained as

$$U_n^*(x) = \{\frac{2x-b-a}{b-a}\}U_{n-1}(x) - U_{n-2}(x), \quad x \geq 2. \tag{16}$$

For a particular case where $x \in [0, 2]$, the shifted forms of (14) and (15) are now given as

$$U_n^*(x) = 2(x-1)U_{n-1}(x) - U_{n-2}(x), \quad n \geq 2. \tag{17a}$$

$$V_{n+1}^*(x) = 2(x-1)V_n(x) - V_{n-1}(x), \quad n \geq 1. \tag{17b}$$

Thus, 12 and 13 now becomes;

$$\langle U_n^*(x), U_m^*(x) \rangle = \int_{0}^{2} \sqrt{2x-x^2} \quad U_n^*(x)U_m^*(x)dx = \begin{cases} 0; & n \neq m, \\ \frac{\pi}{2}; & n = m, \end{cases} \tag{18a}$$

$$\langle V_n^*(x), V_m^*(x) \rangle = \int_{0}^{2} \sqrt{\frac{x}{2-x}} \quad V_n^*(x)V_m^*(x)dx = \begin{cases} 0; & n \neq m, \\ \pi; & n = m, \end{cases} \tag{18b}$$

with the corresponding weight functions given as $(2x-x^2)^{\frac{1}{2}}$ and $(\frac{x}{2-x})^{\frac{1}{2}}$ respectively.
The analytical form of the second kind shifted Chebyshev polynomials of degree n in x denoted by $U_n^*(x)$ is given by

$$U_n^*(x) = \sum_{k=0}^{n} (-1)^k 2^{n-k} \frac{\Gamma(2n-k+2)}{\Gamma(k+1)\Gamma(2n-2k+2)} x^{n-k}, \quad n \geq 0. \tag{19a}$$

The analytical form of the third kind shifted Chebyshev polynomials of degree n in x denoted by $V_n^*(x)$ is given by

$$V_n^*(x) = \sum_{k=0}^{n} (-1)^k 2^{n-k} \frac{(2n+1)\Gamma(2n-k+1)}{\Gamma(k+1)\Gamma(2n-2k+2)} x^{n-k}, \quad n \geq 0. \tag{19b}$$

The function $\omega(x)$ which will be required in the solution of (1) can be written as series of $U_n^*(x)$ and $V_n^*(x)$ respectively as

$$\omega(x) = \sum_{i=0}^{\infty} a_i U_i^*(x) \quad and \quad \omega(x) = \sum_{i=0}^{\infty} a_i V_i^*(x)$$

where the coefficients a_i; $i = 0, 1, 2, ...$ are generated respectively by

$$a_i = \frac{2}{\pi}\int_0^2 \omega(x)\sqrt{2x - x^2}U_i^*(x)dx. \tag{20a}$$

$$a_i = \frac{1}{\pi}\int_0^2 \omega(x)\sqrt{\frac{x}{2-x}}V_i^*(x)dx. \tag{20b}$$

In practice, only the $(n+1)$-terms of shifted Chebyshev polynomials of the second and third kinds are required in the approximation. Thus, we have

$$\omega_n(x) = \sum_{i=0}^{n} a_i U_i^*(x) \tag{21a}$$

$$\omega_n(x) = \sum_{i=0}^{n} a_i V_i^*(x) \tag{21b}$$

4 Solution of the Fractional Derivative Using Shifted Chebyshev Polynomials of the Second and Third Kinds

Suppose $\omega(x)$ is approximated with the help of $(19a)$ and $(19b)$, and taken $\theta > 0$, combining this with the linearity property of Caputo differential operator, we get

$$\omega_n(x) = \sum_{i=0}^{n} a_i D^\theta (U_i^*(x)) \tag{22a}$$

$$\omega_n(x) = \sum_{i=0}^{n} a_i D^\theta (V_i^*(x)) \tag{22b}$$

From the Caputo derivative given in (11), it follows that;

$$D^\theta \left(U_i^*(x)\right) = 0, \quad \text{for} \quad i = 0, 1, \cdots, \lceil\theta\rceil - 1, \quad \theta > 0,$$

$$D^\theta \left(V_i^*(x)\right) = 0, \quad \text{for} \quad i = 0, 1, \cdots, \lceil\theta\rceil - 1, \quad \theta > 0,$$

And

$$D^\theta \left(U_i^*(x)\right) = \sum_{k=0}^{i}(-1)^k 2^{(i-k)} \frac{\Gamma(2i-k+2)}{\Gamma(k+1)\Gamma(2i-2k+2)} D^\theta x^{i-k}. \tag{23a}$$

$$D^\theta \left(V_i^*(x)\right) = \sum_{k=0}^{i}(-1)^k 2^{(i-k)} \frac{(2i+1)\Gamma(2i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+2)} D^\theta x^{i-k}. \tag{23b}$$

Using (11), (23a) and (23b) becomes

$$D^\theta \left(U_i^*(x)\right) = \sum_{k=0}^{i-\lceil\theta\rceil}(-1)^k 2^{(i-k)} \frac{\Gamma(2i-k+2)\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+2)\Gamma(i+1-k-\theta)} x^{i-k-\theta} \tag{24a}$$

$$D^\theta \left(V_i^*(x)\right) = \sum_{k=0}^{i-\lceil\theta\rceil}(-1)^k 2^{(i-k)} \frac{(2i+1)\Gamma(2i-k+1)\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+2)\Gamma(i+1-k-\theta)} x^{i-k-\theta}. \tag{24b}$$

Combining $(22a)$ and $(24a)$, $(22b)$ and $(24b)$, we obtained respectively:

$$D^{\theta}\left(\omega_n(x)\right)=\sum_{i=\lceil\theta\rceil}^{n}\sum_{k=0}^{i-\lceil\theta\rceil}a_i(-1)^k2^{(i-k)}\frac{\Gamma(2i-k+2)\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+2)\Gamma(i+1-k-\theta)}x^{i-k-\theta} \tag{25a}$$

$$D^{\theta}\left(\omega_n(x)\right)=\sum_{i=\lceil\theta\rceil}^{n}\sum_{k=0}^{i-\lceil\theta\rceil}a_i(-1)^k2^{(i-k)}\frac{(2i+1)\Gamma(2i-k+1)\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+2)\Gamma(i+1-k-\theta)}x^{i-k-\theta} \tag{25b}$$

$(25a)$ and $(25b)$ can both be respectively written as:

$$D^{\theta}\left(\omega_n(x)\right)=\sum_{i=\lceil\theta\rceil}^{n}\sum_{k=0}^{i-\lceil\theta\rceil}a_iM_{i,k}^{(\theta)}x^{i-k-\theta}, \tag{26a}$$

$$D^{\theta}\left(\omega_n(x)\right)=\sum_{i=\lceil\theta\rceil}^{n}\sum_{k=0}^{i-\lceil\theta\rceil}a_iG_{i,k}^{(\theta)}x^{i-k-\theta}, \tag{26b}$$

where

$$M_{i,k}^{(\theta)}=(-1)^k2^{(i-k)}\frac{\Gamma(2i-k+2)\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+2)\Gamma(i+1-k-\theta)}. \tag{27a}$$

and

$$G_{i,k}^{(\theta)}=(-1)^k2^{(i-k)}\frac{(2i+1)\Gamma(2i-k+2)\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+2)\Gamma(i+1-k-\theta)}. \tag{27b}$$

The accuracy of our proposed method can be tested as follows:
Suppose $\omega(x)=x^3, n=4$ and $\theta=1.8$
Using the definition of Caputo derivative given in (11), we obtained

$$D^{1.8}x^3=\frac{\Gamma(3+1)}{\Gamma(3+1-1.8)}x^{3-1.8}=\frac{\Gamma(4)}{\Gamma(\frac{11}{5})}x^{\frac{6}{5}}. \tag{28}$$

Now, by using our proposed method given in $(26a)$ and $(27a)$, we have

$$D^{1.8}x^3=\sum_{i=2}^{4}\sum_{k=0}^{i-2}a_iM_{i,k}^{(1.8)}x^{i-k-1.8} \tag{29a}$$

$$=a_2M_{2,0}^{(\frac{9}{5})}x^{\frac{1}{5}}+a_3M_{3,0}^{(\frac{9}{5})}x^{\frac{6}{5}}+a_3M_{3,1}^{(\frac{9}{5})}x^{\frac{1}{5}}+a_4M_{4,0}^{(\frac{9}{5})}x^{\frac{11}{5}}a_4M_{4,1}^{(\frac{9}{5})}x^{\frac{6}{5}}+a_4M_{4,2}^{(\frac{9}{5})}x^{\frac{1}{5}}, \tag{30a}$$

where
$M_{2,0}^{(\frac{9}{5})}=\frac{8}{\Gamma(\frac{6}{5})}$, $M_{3,0}^{(\frac{9}{5})}=\frac{48}{\Gamma(\frac{11}{5})}$, $M_{3,1}^{(\frac{9}{5})}=-\frac{48}{\Gamma(\frac{6}{5})}$, $M_{4,0}^{(\frac{9}{5})}=\frac{384}{\Gamma(\frac{16}{5})}$, $M_{4,1}^{(\frac{9}{5})}=-\frac{384}{\Gamma(\frac{11}{5})}$ and $M_{4,2}^{(\frac{9}{5})}=\frac{168}{\Gamma(\frac{6}{5})}$.
We obtain the constants a_2, a_3 and a_4 using $(20a)$ which are then substituted back into $(30a)$ to have

$$D^{1.8}x^3=\frac{\Gamma(3+1)}{\Gamma(3+1-1.8)}x^{3-1.8}=\frac{\Gamma(4)}{\Gamma(\frac{11}{5})}x^{\frac{6}{5}}. \tag{31a}$$

Since $a_2=\frac{3}{4}$, $a_3=\frac{1}{8}$ and $a_4=0$
In a similar manner, using $(26b)$ and $(27b)$, we have:

$$D^{1.8}x^3=\sum_{i=2}^{4}\sum_{k=0}^{i-2}a_iG_{i,k}^{(1.8)}x^{i-k-1.8} \tag{29b}$$

$$=a_2G_{2,0}^{(\frac{9}{5})}x^{\frac{1}{5}}+a_3G_{3,0}^{(\frac{9}{5})}x^{\frac{6}{5}}+a_3G_{3,1}^{(\frac{9}{5})}x^{\frac{1}{5}}+a_4G_{4,0}^{(\frac{9}{5})}x^{\frac{11}{5}}a_4G_{4,1}^{(\frac{9}{5})}x^{\frac{6}{5}}+a_4G_{4,2}^{(\frac{9}{5})}x^{\frac{1}{5}}, \tag{30b}$$

where
$G_{2,0}^{(\frac{9}{5})}=\frac{8}{\Gamma(\frac{6}{5})}$, $G_{3,0}^{(\frac{9}{5})}=\frac{48}{\Gamma(\frac{11}{5})}$, $G_{3,1}^{(\frac{9}{5})}=-\frac{56}{\Gamma(\frac{6}{5})}$, $G_{4,0}^{(\frac{9}{5})}=\frac{384}{\Gamma(\frac{16}{5})}$, $G_{4,1}^{(\frac{9}{5})}=-\frac{432}{\Gamma(\frac{11}{5})}$ and $G_{4,2}^{(\frac{9}{5})}=\frac{216}{\Gamma(\frac{6}{5})}$.
We obtain the constants a_2, a_3 and a_4 using $(20b)$ which are then substituted back into $(30b)$ to have

$$D^{1.8}x^3=\frac{\Gamma(3+1)}{\Gamma(3+1-1.8)}x^{3-1.8}=\frac{\Gamma(4)}{\Gamma(\frac{11}{5})}x^{\frac{6}{5}}. \tag{31b}$$

Since $a_2=\frac{7}{8}$, $a_3=\frac{1}{8}$ and $a_4=0$

5 Methodology

Consider the given fractional order wave equation (1) together with the conditions (2) and (3). To implement Chebyshev collocation method, we shall let

$$\omega_n(x,t)=\sum_{i=0}^{n}\omega_i(t)U_i^*(x). \tag{32a}$$

and

$$\omega_n(x,t)=\sum_{i=0}^{n}\omega_i(t)V_i^*(x). \tag{32b}$$

Represent the required approximations of second and third kinds Chebyshev polynomials respectively. From (1), $(26a)$ and $(27a)$, it follows that

$$\sum_{i=0}^{n}\frac{d^2\omega_i(t)}{dt^2}U_i^*(x)=\xi(x,t)\sum_{i=\lceil\theta\rceil}^{n}\sum_{k=0}^{i-\lceil\theta\rceil}\omega_i(t)M_{i,k}^{(\theta)}x^{i-k-\theta}+\eta(x,t) \tag{33a}$$

In a similar manner, from (1), $(26b)$ and $(27b)$, we equally get

$$\sum_{i=0}^{n}\frac{d^2\omega_i(t)}{dt^2}V_i^*(x)=\xi(x,t)\sum_{i=\lceil\theta\rceil}^{n}\sum_{k=0}^{i-\lceil\theta\rceil}\omega_i(t)G_{i,k}^{(\theta)}x^{i-k-\theta}+\eta(x,t). \tag{33b}$$

We shall now collocate $(33a)$ and $(33b)$ at the points $x_d=(n+1-\lceil\alpha\rceil)\,,\quad d=0,1,\cdots,n-\lceil\theta\rceil$ as

$$\sum_{i=0}^{n}\ddot{\omega}_i(t)U_i^*(x_d)=\xi(x_d,t)\sum_{i=\lceil\theta\rceil}^{n}\sum_{k=0}^{i-\lceil\theta\rceil}\omega_i(t)M_{i,k}^{(\theta)}x_d^{i-k-\theta}+\eta(x_d,t), \tag{34a}$$

and

$$\sum_{i=0}^{n}\ddot{\omega}_i(t)V_i^*(x_d)=\xi(x_d,t)\sum_{i=\lceil\theta\rceil}^{n}\sum_{k=0}^{i-\lceil\theta\rceil}\omega_i(t)G_{i,k}^{(\theta)}x_d^{i-k-\theta}+\eta(x_d,t), \tag{34b}$$

respectively with $\ddot{\omega}(t)=\frac{d^2\omega}{dt^2}$.
For us to have the best choice of collocation points, we use the roots of shifted Chebyshev polynomial equations $U_{n+1-\lceil\theta\rceil}^*(x)=0$ and $V_{n+1-\lceil\alpha\rceil}^*(x)=0$. Applying the initial condition on $(32a)$ and $(32b)$, and using $(20a)$ and $(20b)$ respectively, we obtained the required constants $\omega_i, i=0,1,...$ at the initial point $t=0$. Furthermore, when the boundary conditions are applied in $(32a)$ and $(32b)$ with $x\in[0,2]$, we get $\lceil\theta\rceil$ equations as follows:

$$\sum_{i=0}^{n}(-1)^i(1+i)\omega_i(t)=0,\quad \sum_{i=0}^{n}(1+i)\omega_i(t)=0 \tag{35a}$$

and

$$\sum_{i=0}^{n}(-1)^{i+2}(1+2i)\omega_i(t)=0,\quad \sum_{i=0}^{n}(-1)^{i}\omega_i(t)=0 \tag{35b}$$

$(34a)$ and $(34b)$ together with the $\lceil\theta\rceil$ equations obtained from the boundary conditions $(35a)$ and $(35b)$ give $(n+1)$-system of second order ordinary differential equations which can be solved by using finite difference method to get the unknown $\omega_i;\ i=0,1,\cdots,n$.

6 Numerical Experiment

Consider the fractional order wave equation of the form

$$\frac{\partial^2\omega(x,t)}{\partial t^2}=\xi(x,t)\frac{\partial^{1.8}\omega(x,t)}{\partial x^{1.8}}+\eta(x,t), \tag{36}$$

defined on a finite domain $0\leq x\leq 2$ and $t>0$ with the coefficient functions

$$\xi(x,t)=\Gamma(1.2)x^{1.8}, \tag{37}$$

and the inhomogeneous source function

$$\eta(x,t)=4e^{-t}x^2(2-x)-16e^{-t}x^2+20e^{-t}x^3, \tag{38}$$

with initial conditions

$$\omega(x,0)=4x^2(2-x),\quad \omega_t(x,0)=-4x^2(2-x) \tag{39}$$

and Dirichlet boundary conditions

$$\omega(0,t)=\omega(2,t)=0. \tag{40}$$

The exact solution for this problem is

$$\omega(x,t)=4e^{-t}x^2(2-x). \tag{41}$$

Solution
We shall consider an approximate solution of the form

$$\omega_n(x,t)=\sum_{i=0}^{n}\omega_i(t)U_i^*(x) \tag{42}$$

for $n=3$, we have

$$\omega_3(x,t)=\sum_{i=0}^{3}\omega_i(t)U_i^*(x) \tag{43}$$

Using (43) in $(34a)$, we get

$$\sum_{i=0}^{3}\frac{d^2\omega_i(t)}{dt^2}U_i^*(x_d)=\xi(x_d,t)\sum_{i=2}^{3}\sum_{k=0}^{i-2}\omega_i(t)M_{i,k}^{(\theta)}x_d^{i-k-\theta}+\eta(x_d,t)\quad d=0,1 \tag{44}$$

Using (44) and $(35a)$, we arrive at the following system of second order ordinary differential equations:

$$\left.\begin{aligned}
\ddot{\omega}_0(t)+c_1\ddot{\omega}_1(t)+c_2\ddot{\omega}_3(t)&=D_1\omega_2(t)+D_2\omega_3(t)+\eta(x_0,t)\\
\ddot{\omega}_0(t)+c_{11}\ddot{\omega}_1(t)+c_{22}\ddot{\omega}_3(t)&=D_{11}\omega_2(t)+D_{22}\omega_3(t)+\eta(x_1,t)\\
\omega_0(t)-\omega_1(t)+\omega_2(t)-\omega_3(t)&=0\\
\omega_0(t)+3\omega_1(t)+5\omega_2(t)+7\omega_3(t)&=0,
\end{aligned}\right\} \tag{45}$$

where
$c_1 = U_1^*(x_0),\ c_2 = U_3^*(x_0), c_{11} = U_1^*(x_1),\ c_{22} = U_3^*(x_1), D_1 = \xi(x_0,t)M_{2,0}^{1.8}x_0^{\frac{1}{5}},$
$D_2 = \xi(x_0,t)\{M_{3,0}^{1.8}x_0^{\frac{6}{5}} + M_{3,1}^{1.8}x_0^{\frac{1}{5}}\},\ D_{11} = \xi(x_1,t)M_{2,0}^{1.8}x_1^{\frac{1}{5}}, D_{22} = \xi(x_1,t)\{M_{3,0}^{1.8}x_1^{\frac{6}{5}} + M_{3,1}^{1.8}x_1^{\frac{1}{5}}\}.$
To solve the system of equations 45 using finite difference method, we will use the notations $t_i = i\Delta t$ for $i = 0, 1, \cdots, N$, with $\Delta t = \frac{T}{N}$ and $T = T_{final}$. we also defined $\omega_i^n = \omega_i(t_n), \quad \eta_i^n = \eta_i(t_n)$,
The system (45) now becomes

$$\begin{aligned}\frac{\omega_0^{n+1} - 2\omega_0^n + \omega_0^{n-1}}{(\Delta t)^2} + c_1\frac{\omega_1^{n+1} - 2\omega_1^n + \omega_1^{n-1}}{(\Delta t)^2} + c_2\frac{\omega_3^{n+1} - 2\omega_3^n + u_3^{n-1}}{(\Delta t)^2}\\ = D_1\omega_2^{n+1} + D_2\omega_3^{n+1} + \eta_0^{n+1} \end{aligned} \tag{46}$$

$$\begin{aligned}\frac{\omega_0^{n+1} - 2\omega_0^n + \omega_0^{n-1}}{(\Delta t)^2} + c_{11}\frac{\omega_1^{n+1} - 2\omega_1^n + \omega_1^{n-1}}{(\Delta t)^2} + c_{22}\frac{\omega_3^{n+1} - 2\omega_3^n + \omega_3^{n-1}}{(\Delta t)^2}\\ = D_{11}\omega_2^{n+1} + D_{22}\omega_3^{n+1} + \eta_1^{n+1} \end{aligned} \tag{47}$$

$$\omega_0^{n+1} - \omega_1^{n+1} + \omega_2^{n+1} - \omega_3^{n+1} = 0 \tag{48}$$

$$\omega_0^{n+1} + 3\omega_1^{n+1} + 5\omega_2^{n+1} + 7\omega_3^{n+1} = 0 \tag{49}$$

(46)-(49) can be written in matrix form as

$$\begin{aligned}&\begin{pmatrix} 1 & c_1 & -\Delta t^2 D_1 & c_2 - \Delta t^2 D_2 \\ 1 & c_{11} & -\Delta t^2 D_{11} & c_{22} - \Delta t^2 D_{22} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}^{n+1} = \begin{pmatrix} 1 & 2c_1 & 0 & 2c_2 \\ 1 & 2c_{11} & 0 & 2c_{22} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}^{n} \\ &- \begin{pmatrix} 1 & c_1 & 0 & c_2 \\ 1 & c_{11} & 0 & c_{22} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}^{n-1} + (\Delta t)^2 \begin{pmatrix} \eta_0 \\ \eta_1 \\ 0 \\ 0 \end{pmatrix}^{n+1} \end{aligned} \tag{50}$$

which is equivalent to

$$E\Omega^{n+1} = F\Omega^n - G\Omega^{n-1} + (\Delta t)^2 H^{n+1} \tag{51a}$$

or

$$\Omega^{n+1} = E^{-1}FU^n - E^{-1}GU^{n-1} + E^{-1}(\Delta t)^2 H^{n+1}, \tag{52a}$$

Following the same process demonstrated in (43)-(50) and using $(34b)$ and $(35b)$ on the same problem given in (36)-(41), we obtained

$$W\Omega^{n+1} = X\Omega^n - Y\Omega^{n-1} + (\Delta t)^2 Z^{n+1} \tag{51b}$$

or

$$\Omega^{n+1} = W^{-1}XU^n - W^{-1}YU^{n-1} + W^{-1}(\Delta t)^2 Z^{n+1}, \tag{52b}$$

where
$\Omega^n = (\omega_0^n, \omega_1^n, \omega_2^n, \omega_3^n)^T$ and $H^n = ((\Delta t)^2\eta_0^n, (\Delta t)^2\eta_1^n, 0, 0)^T$
To obtain the initial solution Ω^0 of $(52a)$ and $52b$, we use the initial condition of the problem $\omega(x, 0)$ combine with $(20a)$ and $20b$ respectively. We shall also obtain Ω^1 of $(52a)$ and $52b$, using the initial condition of the problem $\omega_t(x, 0)$ as follows

$$\omega_t(x,0) = \frac{\omega_{i,1} - \omega_{i,0}}{\Delta t}, \quad i = 0, 1, ... \tag{53}$$

then

$$\omega_{i,1} = \omega_{i,0} + \omega^{'}(x_i)\Delta t, \quad i = 0, 1, ... \tag{54}$$

hence

$$\Omega^1 = \Omega^0 + \Delta t R, \tag{55}$$

where

$$R = \omega^{'}(x_i). \tag{56}$$

The approximate solution is then obtained by substituting the results derived above in the series (43).

7 Results, Discussion of Results and Conclusion

7.1 Results

Table 1: Results obtained using $U_n(x)$ and $V_n(x)$ for $n = 3$, $T = 1$ and $\Delta t = \frac{1}{400}$

x	Exact sol.	Approx. sol.($U_n(x)$)	Approx. sol.($V_n(x)$)	Approx. sol. [16]	Error($U_n(x)$)	Error($V_n(x)$)	Error[16]
0.00	0	0	0	0	0	0	0
0.20	0.2872808993	0.26861000911	0.2596317122	0.2853666054	1.667089×10^{-2}	2.764919×10^{-2}	1.914294×10^{-3}
0.40	1.0188927790	0.99784927790	0.9820922350	0.9747779627	2.104610×10^{-2}	3.680054×10^{-2}	4.411482×10^{-2}
0.60	2.0009365580	1.9925884911	1.9721847885	1.9862317000	1.348067×10^{-2}	2.875170×10^{-2}	1.470486×10^{-2}
0.80	3.0414330890	3.0577193060	3.0347125926	3.1096670040	1.628622×10^{-2}	1.720496×10^{-2}	6.823392×10^{-2}
1.00	3.9503112020	3.9981193981	3.9744788867	4.0900055000	4.780820×10^{-2}	2.416768×10^{-2}	1.396943×10^{-1}
1.20	4.5395818000	4.6186699383	4.5962868328	4.6852897270	7.908813×10^{-2}	5.670503×10^{-2}	1.457079×10^{-1}
1.40	4.6223961170	4.7242520977	4.7049397088	4.6992014000	1.018560×10^{-1}	8.254359×10^{-2}	7.752402×10^{-2}
1.60	4.0148937660	4.1197470471	4.1052407153	3.9929906310	1.048533×10^{-1}	9.034695×10^{-2}	2.190314×10^{-2}
1.80	2.5343312070	2.6100359575	2.6019930724	2.4617225700	7.570475×10^{-2}	6.766187×10^{-2}	7.260864×10^{-2}
2.00	0	0	0	0	0	0	0

Table 2: Results obtained using $U_n(x)$ and $V_n(x)$ for $n = 3$, $T = 1$ and $\Delta t = \frac{1}{4000}$

x	Exact sol.	Approx. sol.($U_n(x)$)	Approx. sol.($V_n(x)$)	Approx. sol. [16]	Error($U_n(x)$)	Error($V_n(x)$)	Error[16]
0.00	0	0	0	0	0	0	0
0.20	0.2872808993	0.2860144777	0.2848944100	0.2863548880	1.266423×10^{-3}	2.386489×10^{-3}	9.260113×10^{-4}
0.40	1.0188927790	1.0213496360	1.0195198087	1.0221455440	2.456857×10^{-3}	6.270297×10^{-3}	3.252965×10^{-3}
0.60	2.0009365580	2.0136756798	2.0114858745	2.0149684560	1.273912×10^{-2}	1.054932×10^{-2}	1.403190×10^{-2}
0.80	3.0414330890	3.0706628142	3.0684022857	3.0724231120	2.922972×10^{-2}	2.696920×10^{-2}	3.099002×10^{-2}
1.00	3.9503112020	3.9999812440	3.9978782089	4.0021090000	4.967004×10^{-2}	4.756701×10^{-2}	5.179780×10^{-2}
1.20	4.5395818000	4.6093011744	4.6075248584	4.6116256080	6.971937×10^{-2}	6.794305×10^{-2}	7.222980×10^{-2}
1.40	4.6223961170	4.7242520977	4.7049397088	4.6992014000	1.018560×10^{-1}	8.254359×10^{-2}	7.752402×10^{-2}
1.60	4.0148937660	4.0986263564	4.0977649540	4.1005489400	8.389669×10^{-2}	8.255426×10^{-2}	8.617630×10^{-2}
1.80	2.5343312070	2.5939720180	2.5939720180	2.5951546300	5.964081×10^{-2}	6.082342×10^{-2}	7.260864×10^{-2}
2.00	0	0	0	0	0	0	0

7.2 Discussion of Results

Tables 7.1 and 7.2 presented above compare the relationship between the exact and approximate solutions obtained using the proposed method for the case $n = 3$ and the result reported in ([16]) at different step sizes Δt= $\frac{1}{400}$ and Δt = $\frac{1}{4000}$ using shifted Chebyshev polynomial of the second and the third kinds. It is obvious that the result in columns 3 and 4 of table 7.2 is better that that of column 3 and 4 of table 7.1. This is based on the general understanding that whenever a differential equation is stable, the smaller the step size, the better the results. Meanwhile, the results in columns 5 of Table 7.1 and Table 7.2 are better than that of columns 3 and 4 of both tables. Also, the results in columns 3 of Table 7.1 and Table 7.2 are better than that of columns 4 of both tables. That is, fourth kind polynomial approximates better than the second kind polynomial and the second kind polynomial, approximates better than the third kind polynomial. The 3D graphs presented are for the exact solution, and the approximate solutions obtained using different step sizes. It is also worthy to note that the accuracy of our results depends on the step-size rather than the order of the given problem.

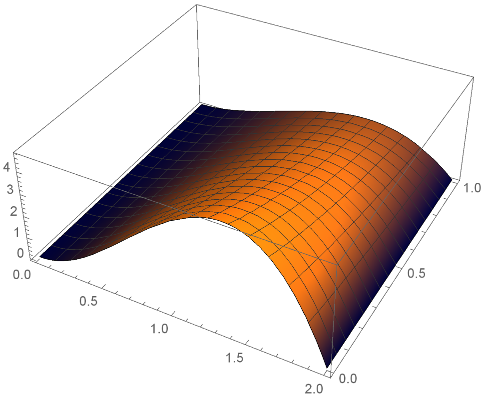

Figure 1: Exact solution graph

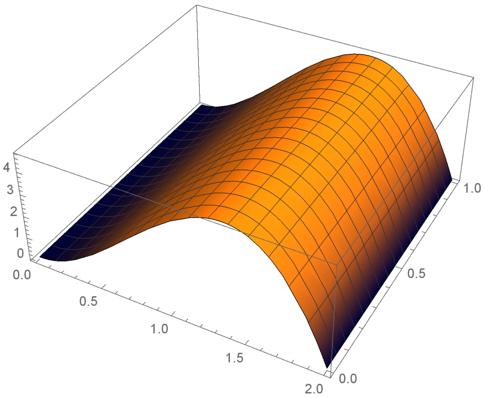

Figure 2: Approximate solution graph $U_n(x)$, for $\Delta t = \frac{1}{400}$

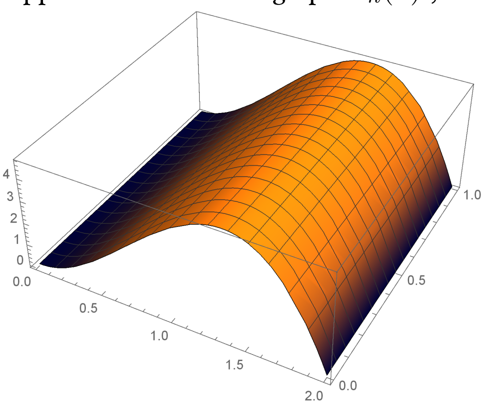

Figure 3: Approximate solution graph $U_n(x)$, for $\Delta t = \frac{1}{4000}$

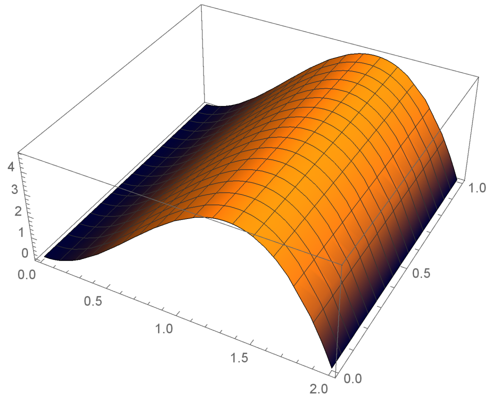

Figure 4: Approximate solution graph $V_n(x)$, for $\Delta t = \frac{1}{400}$

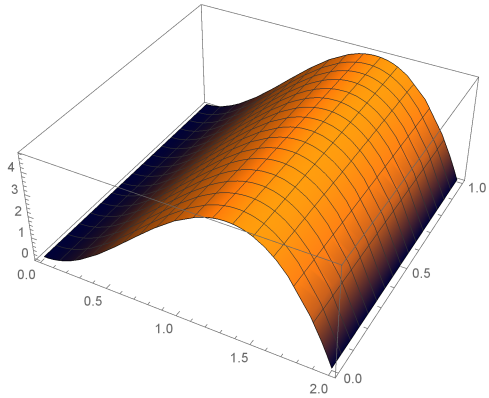

Figure 5: Approximate solution graph $V_n(x)$, for $\Delta t = \frac{1}{4000}$

7.3 Conclusion

This research paper talks about the numerical solution of SFOWE with the fractional derivate expressed in Caputo sense, and the given fractional order PDE were reduced to system of second order ODEs which are later transformed to a system of linear algebraic equations with the aid of finite difference method, and later collocated at the roots of the shifted Chebyshev polynomials of the second and third kinds respectively. The approximate solution obtained using the proposed scheme is compared with the exact solution, and it was realized that the proposed scheme is very reliable and accurate. All computations in this work are carried out using Mathematica 11.0.

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