Journal of the Nigerian Society for Mathematical Biology
Volume 2 (2019), pp. 41 – 68

Mathematical model for the transmission dynamics of tuberculosis with passive immunity, drug sensitive tuberculosis and drug-resistant tuberculosis

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Abstract

This study presents a mathematical model for the transmission dynamics of tuberculosis in the drug sensitive TB and drug-resistant TB. Eight (8) compartments were considered, namely: passively immune infants, susceptible, latently effected with DS-TB, latently infected with DR-TB, infectious with DS-TB, infectious with DR-TB, recovered with DS-TB and recovered with DR-TB and we mathematically modelled the natural growth, the interactions between these populations and the effects of treatments. The disease-free equilibrium (DFE) and endemic equilibrium (EE) were obtained. We obtained the basic reproduction number, cwhich can be used to control the transmission dynamics of the disease and thus, established the conditions for local and global stability of the disease free- equilibrium using Routh- Hurwitz criterion using Routh-Hurwitz theorem and Lasalle-Lyapunov function respectively. The result of the analysis of the stability of the disease-free equilibrium state that tuberculosis can totally be eradicated if effort is made to ensure that the rate of recovery infected individuals with DS-TB and DR-TB and the rate of natural death must have a lower bound. Numerical analysis for the model was carried out and it demonstrated that in the case of patients with both DS-TB and DR-TB, tuberculosis will be eradicated if effort is intensified in bringing down both the transmission rate of DS-TB and DR-TB through the using of BCG vaccine and continuous treatment.

Keywords: Passive immunity; Drug Sensitive TB, Drug resistant TB; Endemic equilibrium; Disease free-equilibrium

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1 Introduction

Tuberculosis (TB) is a bacterial disease acquired through air-borne infection. Mycobacterium Tuberculosis Complex (MTBC) is the causative agent of tuberculosis. According to the World Health Organization (WHO), one-third of the world's population is infected, either latently or active, with tuberculosis [19]. It is an ancient disease with evidence of its existence being found in relics from ancient Egypt, India and China. In the eighteenth century, Western Europe suffered terribly from this disease with prevalence as high as 900 deaths per 100,000. This was largely due to poor ventilation, overcrowded housing, primitive sanitation and malnutrition among other risk factors that led to the epidemic [6].

Tuberculosis is spread through the air from one person to another. The bacteria get into the air when someone who has a tuberculosis lung infection coughs, sneezes, shouts, or spits. People who are nearby can then possibly breathe the bacteria into their lungs and become infected. Even though the disease is airborne, it is believed that TB is not highly infectious and so, occasional contacts with infectious person rarely lead to infection. TB cannot be spread through handshakes, sitting on toilet seats or sharing dishes and utensils with someone who has TB [1].

TB is the ninth leading cause of death worldwide and the leading cause from a single infectious agent, ranking above HIV/AIDS [20]. In 2016, there were an estimated 1.3 million TB deaths among HIV-negative people (down from 1.7 million in 2000) and an additional 374,000 deaths among HIV-positive people. An estimated 10.4 million people fell ill with TB in 2016: 90% were adults, 65% were male, 10% were people living with HIV (74% in Africa) and 56% were in five countries: India, Indonesia, China, the Philippines and Pakistan [20]. Drug-Resistant TB is a continuing threat, in 2016, there were 600, 000 new cases with Resistance to Rifampicin (RR-TB), the most effective first-line drug, of which 490, 000 had Multidrug-Resistant TB (MDR-TB). Almost half (47%) of these cases were in India, China and the Russian Federation [20].

Globally, the TB mortality rate is falling at about 3% per year. TB incidence is falling at about 2% per year and 16% of TB cases die from the disease; by 2020, these figures need to improve to 4–5% per year and 10%, respectively, to reach the first (2020) milestones of the End TB Strategy [20]. Most deaths from TB could be prevented with early diagnosis and appropriate treatment. Millions of people are diagnosed and successfully treated for TB each year, averting millions of deaths (53 million 2000–2016), but there are still large gaps in detection and treatment. In 2016, 6.3 million new cases of TB were reported (up from 6.1 million in 2015), equivalent to 61% of the estimated incidence of 10.4 million; the latest treatment outcome data show a global treatment success rate of 83%, similar to recent years [20]. There were 476, 774 reported cases of HIV-positive TB (46% of the estimated incidence), of whom 85% were on antiretroviral therapy (ART). A total of 129, 689 people were started on treatment for Drug-Resistant TB, a small increase from 125, 629 in 2015 but only 22% of the estimated incidence; treatment success remains low, at 54% globally [20].

According to the 2016 Global TB report, Nigeria has the highest TB burden in Africa and ranked 4th in the world. It is among the six countries that accounted for 60% of the global burden of TB [20]. They further revealed that Nigeria and India accounted for 48% of global TB deaths among HIV-negative people and for 43% of the combine total TB deaths in HIV-negative and HIV-positive. The same report also revealed that Nigeria accounted for 77% of the global gap in TB case finding. In 2016, Nigeria notified less than 20% of the total TB cases estimated for that year [20]. Therefore,

more than 80% TB cases in the country are undiagnosed TB cases in the community which serve as a reservoir for continuing transmission of TB.

Psychosocial distress that communities go through is enormous. This involves thinking about the loss of their loved ones and the economic impact of taking care of the sick ones especially among the low-income individuals. This impacts not only the individuals, but also the economic progress of the country. Over the last twenty-five years, the mortality rate of tuberculosis has greatly decreased by forty-five percent since and this is largely due to effective diagnosis and treatment [16]. The symptoms of TB are usually mild and tend to present over a period of weeks, months, or sometimes years. TB disease symptoms are often initially mistaken for a smoker's cough, allergies, or chronic bronchitis from a lingering cold or flu infection. TB infection most often affects the lungs but can cause problems in other parts of the body. The classic symptoms of TB in the lungs include: cough lasting more than three weeks, unexplained weight loss, low-grade fever, and night sweats [5].

Vaccination is the administration of weakened antigens to produce immunity to a disease. Prevention of tuberculosis relies on screening programs and vaccination usually with Bacillus Calmette Guerin (BCG) vaccine [14]. While the vaccine protects against severe form of TB in children (TB meningitis and Miliary TB), for the meantime, a vaccine that is effective in preventing TB in adults remains elusive; poor administration of this vaccine and environmental conditions can make the vaccines inefficient. Tuberculosis diagnosis relies on radiology (commonly chest X-ray), a tuberculin skin test, blood tests as well as microscopic examination and microbiological culture of body fluid such as sputum. Tuberculosis usually attacks the lungs but can also affect the spine, bones, central nervous system and even the skin [16].

Treatment for Tuberculosis uses antibiotics and requires much longer period of treatment (around 6 to 24 months) to entirely eliminate Mycobacterium from the body. The Directly Observed Treatment Shorts (DOTS) strategy as recommended by WHO makes sure diagnosis and medicine are available for all TB patients free of charge and has helped in the control and management of tuberculosis [20].

There is an emerging form of tuberculosis commonly known as Multi-Drug Resistant (MDR) tuberculosis which is defined as tuberculosis resistant to both of the two most effective first line of antibiotic treatment of active tuberculosis *i.e.*, Isoniazid (INH) and Rifampin (RIF), and it is harder and more expensive to treat. It is currently a major health concern to medical workers and researchers and one can get MDR tuberculosis by either spending time with an MDR patient and breathing in the MDR tuberculosis bacteria or those with active tuberculosis not following their prescribed treatment regimen or TB medicine not being readily available to them. MDR tuberculosis is much more difficult to treat and the mortality of persons with this form of tuberculosis is far much higher if the second line of antibiotic treatment is not affected promptly [4].
In 2016, there were an estimated 480, 000 new cases of Multidrug-Resistant TB (MDR-TB) and an additional 100, 000 people with Rifampicin-Resistant TB (RR-TB) who were also newly eligible for MDR-TB treatment. India, China and the Russian Federation accounted for 45% of the combined total of 580, 000 cases [20].

Mathematical models of TB have played a key role in the formulation of tuberculosis prevention and control strategies and establishment of interim goals for intervention programmes. These models are based on the underlying transmission mechanism of TB to help public heath administrative workers and our stakeholders in TB control understand better how the disease is spread. At this juncture, we review some past and recent works on tuberculosis. The work in [18] developed the first deterministic mathematical model to study the epidemiology of tuberculosis for the transmission dynamics of tuberculosis. This shows that with time other models have been developed to help prevent the risk of transmission of tuberculosis. The work in [8] developed mathematical model for dynamics of TB disease with vaccination, taking into consideration the Passively Immune Infants (PII) and the vaccination of the susceptible. They considered a Susceptible-Exposed-Infectious-Recovered (SEIR) model by introduced the passively immune infants resulting to an MSEIR model. The dynamics of the compartments were described by system of ordinary differential equations which were solved algebraically, and analyzed for stability. It was established that the disease free equilibrium state of the model is stable, when the basic reproduction number, $R_0 < 1$, it was also established that the endemic states for the modified model is stable using Bellman and Cooke theorem and that if efforts are made to ensure that more susceptible infants are vaccinated, the breakdown of the susceptible and progression to infectious state is reduced.

The work in [2] used a deterministic model with isolation where they studied the transmission dynamics of three strains of Mycobacterium Tuberculosis (TB), namely; the Drug-Sensitive, Multi-Drug-Resistance (MDR) and Extensively Drug-Resistance (XDR) tuberculosis strains. Their result of the global sensitivity analysis indicated that the dominant parameters are the disease progression rate, the recovery rate, the infectivity parameter, the isolation rate, and the rate of cost to follow up and fraction of fast progression rates. They also found that an increase in isolation rate leads to a decrease in the total number of individuals who are cost to follow up. The work in [17] used a mathematical model to simulate tuberculosis transmission in the highly endemic regions of the Asia-pacific. They found out that their model could not be calibrated to the estimated incidence rate without allowing for re-infection during latency and that even in the presence of a moderate fitness cost and a lower value of R_0, MDR tuberculosis becomes the dominant strain at equilibrium.

Improved treatment of Drug-Susceptible tuberculosis did not result in decreased rates of MDR tuberculosis through prevention of the new Resistance but rather resulted in a modest increase in MDR tuberculosis. [11] formulated a model for global stability of the endemic equilibrium of tuberculosis with immigration and treatment. In their work, immigration was considered and ended up concluding that constant influx of latent TB can launch a TB epidemic itself irrespective of the initial conditions. And the result simulation shows that a higher level of latent immigrants will produce a higher level of TB incidence. The work in [9] presented a work on Modelling Socio-demography to capture tuberculosis transmission dynamics in a low burden setting, age and work place were considered as factors of tuberculosis dynamics. Their result shows that the higher the age, the higher the risk of being infected and also the larger the population of a workplace, the smaller the percentage of infection. They concluded that lower infection rate has been suggested for a workplace of larger size; the majority of workplace with infection rate beyond 30% has less than six employees. The work in [21] used an SEIR-model with fast and slow progression to numerically compare the effects of preventative treatment of those in the fast-progressing latent class with treatment of those with active, infectious disease. They concluded that contact tracing and preventative treatment compare quite favorably to treatment of those with the disease.

The work in [15] presented a model with 19 different states, which includes fast and slow progression, treatment with chemoprophylaxis (INH), super infection (e.g. reinfection), three clinical categories of active disease, and "good" and "bad" treatment. The authors calibrated the model to five different regions of the world, and used it (numerically) to quantitatively predict that a major extension to Directly Observed Therapy Short-Course (DOTS) could be quite effective, preventing millions of deaths. The work in [7] formulated a model that allowed the relative fitness of Drug-Resistant strains to be as high as (i.e. the same as drug-sensitive strains), and concluded that Drug-Resistant strains could threaten control of TB. Minimally, they estimated that 70% of Drug-Resistant cases must be detected and 80% of these must be cured in order to prevent Drug-Resistant TB outbreaks while [10] formulated a mathematical model on dynamics of mycobacterium tuberculosis to assess the impact of TB diagnosis and treatment. Qualitative analysis was done and analysis shows two equilibrium points of the model. The disease free-equilibrium which is locally and globally asymptotically stable for $R_0 < 1,$ and the endemic equilibrium that is locally asymptotically stable for $R_0 < 1.$ Numerical analysis was done and results suggested that as a proportion of TB patients being presented to clinics for diagnosis have increased, the rate of treatment should be correlated to the number of diagnosed infected individuals. But also, diagnosis and treatment can cause psychological stress because of the processes involved. The work in [13] modeled the effect of vaccination and treatment on the transmission dynamics of Tuberculosis (TB). The analysis of the Endemic equilibrium state of the model, using the Basic Reproduction number, R_0 shows that TB can effectively be controlled or even be eradicated if effort is made to ensure that the total removal rate from both the Latent and the Infectious classes is always less than the product of total contraction and total breakdown of the Susceptible class.

2 Model formulation

In this section we will develop a deterministic model in which the individuals in the population are assigned to different compartments, each representing a specific stage of the epidemic. The model to be considered is a Susceptible-Exposed-Infected-Recovered (SEIR) model which considers the passively immune infants M which will help us understand the dynamics of tuberculosis transmission.

2.1 Assumption of the model

The model is based on the following assumptions:

i. The population is heterogeneous. That is, the individuals that make up the population can be grouped into different compartments or classes according to their epidemiological state.
ii. It is assumed that the only way of entry into the population is through birth of new born babies and the only way of exit is via death from natural causes or death from TB related causes.
iii. All newborns are previously uninfected by TB and therefore join either immunized compartment or the susceptible compartment depending on whether they are vaccinated or not.
iv. The vaccinated individuals do not acquire permanent immunity.

v. The effective treatment rate of Drug-Resistant TB is lower than that of Drug-Sensitive TB.

Variables	Description
$M(t)$	the number of individuals who are immunized against TB through vaccination at time t.
$S(t)$	the number of susceptible individuals at time t.
$E_S(t)$	the number of latently infected individuals with Drug-Sensitive TB at time t.
$E_R(t)$	the number of latently infected individuals with Drug-Resistant at time t.
$I_S(t)$	the number of infectious individuals with Drug-Sensitive TB at time t.
$I_R(t)$	the number of infectious individuals with Drug-Resistant TB at time t.
$R_S(t)$	the number of recovered individuals with Drug-Sensitive TB at time t.
$R_R(t)$	the number of recovered individuals with Drug-Resistant TB at time t.
β_S	the transmission rate of Drug-Sensitive TB
β_R	the transmission rate of Drug-Resistant TB
π	the recruitment rate
r_1	the treatment efficacy of Drug-Sensitive TB
r_2	the treatment efficacy of Drug-Resistant TB
σ	proportion of new births that have been immunized through vaccination
θ	the rate of expiration of vaccine efficacy
r	the probability of Drug-Resistant TB emerging during treatment
μ	natural mortality rate
υ	progression rate from latent TB to active TB for both DS-TB and DR-TB cases
μ_T	mortality rate due to TB
ρ_S	the proportion of new infections that produce active TB for DS-TB cases
ρ_R	the proportion of new infections that produce active TB for DS-TB case

Table 1. Variables and parameters of the model with passive immunity, drug-sensitive TB and drug-resistant TB

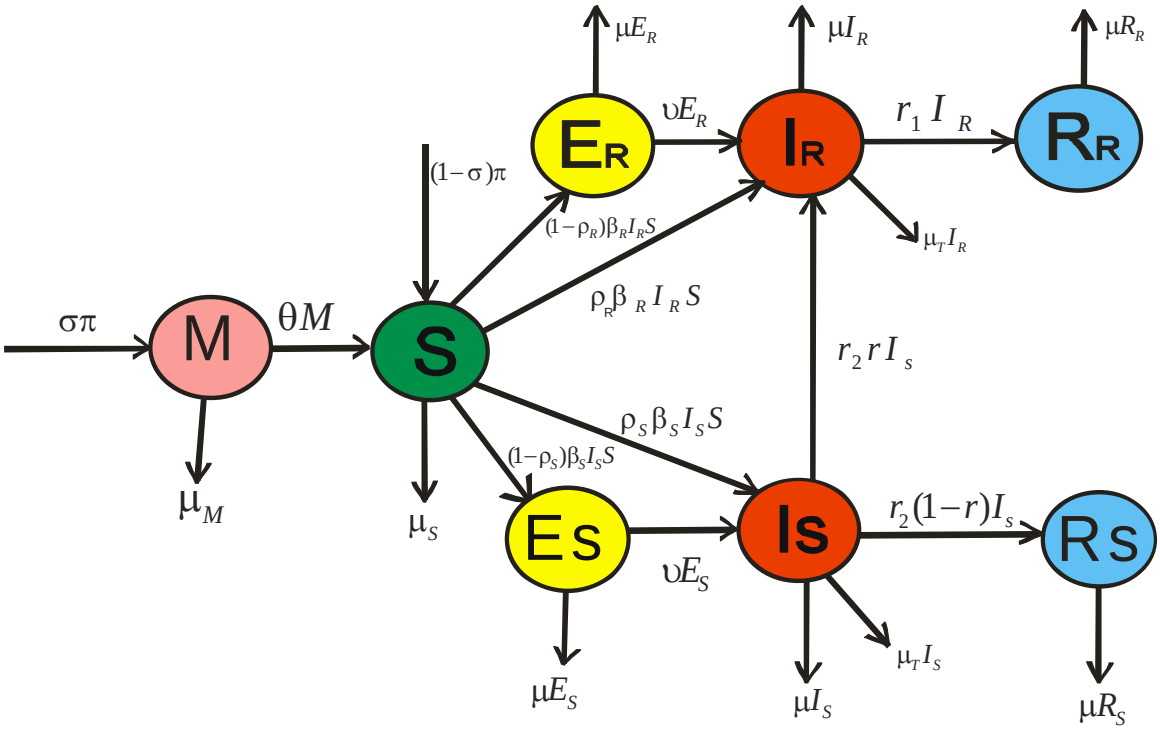

Figure 1. Schematic diagram for the model with passive immunity, drug-sensitive TB and drug-resistant TB

2.2 Model equations

$$\frac{dM}{dt} = \sigma\pi - (\theta + \mu)M \tag{1}$$

$$\frac{dS}{dt} = (1-\sigma)\pi + \theta M - (\beta_S I_S + \beta_R I_R + \mu)S \tag{2}$$

$$\frac{dE_S}{dt} = (1-\rho_S)\beta_S I_S S - (\upsilon + \mu)E_S \tag{3}$$

$$\frac{dE_R}{dt} = (1-\rho_R)\beta_R I_R S - (\upsilon + \mu)E_R \tag{4}$$

$$\frac{dI_S}{dt} = \rho_S\beta_S I_S S + \upsilon E_S - (\mu + \mu_T + r_2)I_S \tag{5}$$

$$\frac{dI_R}{dt} = \rho_R \beta_R I_R S + \upsilon E_R + r_2 r I_S - (\mu + \mu_T + r_1) I_R \quad (6)$$

$$\frac{dR_S}{dt} = r_2(1-r) I_S - \mu R_S \quad (7)$$

$$\frac{dR_R}{dt} = r_1 I_R - \mu R_R \quad (8)$$

where $\left.\begin{array}{l} M(0) = M_0, S(0) = S_0, E_S(0) = E_{S_0}, E_R(0) = E_{R_0}, \\ I_S(0) = I_{S_0}, I_R(0) = I_{R_0}, R_R(0) = R_{R_0}, R_R(0) = R_{R_0}. \end{array}\right\} (z)$

Note that $N = M + S + E_S + E_R + I_S + I_R + R_S + R_R$.

3 Model Analysis

In this section, the conditions for the existence of equilibria of the system (1) – (8) are explored.

3.1 Existence of invariant region

From the model equations (1)- (8), the total population is given by

$N = M + S + E_S + E_R + I_S + I_R + R_S + R_R$

That is $\frac{dN}{dt} = \frac{dM}{dt} + \frac{dS}{dt} + \frac{dE_S}{dt} + \frac{dE_R}{dt} + \frac{dI_S}{dt} + \frac{dI_R}{dt} + \frac{dR_S}{dt} + \frac{dR_R}{dt}$

Therefore, adding the differential equations, we have

$$\frac{dN}{dt} = \pi - \mu(M + S + E_S + E_R + I_S + I_R + R_S + R_R) - \mu_T(I_S + I_R)$$

$$\frac{dN}{dt} = \pi - \mu N - \mu_T(I_S + I_R)$$

$$\frac{dN}{dt} \leq \pi - \mu N$$

By method of integration

$$N(t) \leq N(0)e^{-\mu t} + \frac{\pi}{\mu}(1 - e^{-\mu t})$$

$$N(t) \leq \frac{\pi}{\mu} + (N(0) - \frac{\pi}{\mu})e^{-\mu t}$$

$$\lim_{i \to \infty} N(t) \leq \frac{\pi}{\mu}$$

As $\lim_{i\to\infty} e^{-\mu t}$

So that at $t \to \infty, N(t) \le \frac{\pi}{\mu}.$

The region in which the model makes biological sense is given by

$$\Omega = \left\{ M, S, E_S, E_R, I_S, I_R, R_S, R_R \in ¡^{8}_{+} : M + S + E_S + E_R + I_S + I_R + R_S + R_R \le \frac{\pi}{\mu} \right\}$$

This means that every solution with initial condition (zs) in Ω remains in Ω for all. Therefore, in the region Ω, our model is biologically feasible, mathematically well posed and positively invariant.

3.2 Disease free equilibrium state

At equilibrium, setting equations (1) – (8) to zero we get

$$\sigma\pi - (\theta + \mu)M = 0 \quad (9)$$

$$(1-\sigma)\pi + \theta M - (\beta_S I_S + \beta_R I_R + \mu)S = 0 \quad (10)$$

$$(1-\rho_S)\beta_S I_S S - (\upsilon + \mu)E_S = 0 \quad (11)$$

$$(1-\rho_R)\beta_R I_R S - (\upsilon + \mu)E_R = 0 \quad (12)$$

$$\rho_S \beta_S I_S S + \upsilon E_S - (\mu + \mu_T + r_2)I_S = 0 \quad (13)$$

$$\rho_R \beta_R I_R S + \upsilon E_R + r_2 r I_S - (\mu + \mu_T + r_1)I_R = 0 \quad (14)$$

$$r_2(1-r)I_S - \mu R_S = 0 \quad (15)$$

$$r_1 I_R - \mu R_R = 0 \quad (16)$$

Solving equations (9) - (16) simultaneously, we obtain the disease -free equilibrium state:

$$E_0 = \left(\frac{\sigma\pi}{\theta + \mu}, \frac{\pi(\theta + \mu - \mu\sigma)}{\mu(\theta + \mu)} 0, 0, 0, 0, 0, 0 \right) \quad (17)$$

3.3 Basic reproduction number

The basic reproduction number for both drug sensitive TB and drug-resistant TB is denoted by R_{0S} and R_{0R} respectively. It is an important parameter that is used to study the behaviour of epidemiological models. It is defined as the average number of secondary infections infected by an infective individual during an infective period provided that the all members of the population are

susceptible. It is an important threshold parameter that determines whether or not, an infection will spread through a given population.

We apply the next generation matrix technique by Diekman and Heesterbeck (2000) to obtain the basic reproduction number, R_{0S} and R_{0R} by considering the drug sensitive infected compartments of the system (1) to (8). That is equations (3), (4), (5) and (6).

Let F_i be the rate of appearance of new infection in the i compartment and V_i be the rate of transfer of individuals out of i, given the disease free equilibrium, then R_0 is the spectral radius (largest Eigen values) of the next generation matrix denoted by $G = \rho FV^{-1}$

3.3.1 Reproduction number for drug-sensitive TB

From equations (3) and (5) we have

$$\frac{dE_S}{dt} = (1-\rho_S)\beta_S I_S S - (\upsilon+\mu)E_S$$
$$\frac{dI_S}{dt} = \rho_S\beta_S I_S S + \upsilon E_S - (\mu+\mu_T+r_2)I_S$$

We evaluated F and V, which are the Jacobian matrices of $\hat{F}$ and $\hat{V}$, respectively at disease free equilibrium E_0 to get

$$F = \begin{bmatrix} 0 & \dfrac{(1-\rho_S)\beta_S\pi(\theta+\mu-\mu\sigma)}{\mu(\theta+\mu)} \\ 0 & \dfrac{\rho_S\beta_S\pi(\theta+\mu-\mu\sigma)}{\mu(\theta+\mu)} \end{bmatrix}, \quad V = \begin{bmatrix} (\upsilon+\mu) & 0 \\ -\upsilon & (\mu+\mu_T+r_2) \end{bmatrix}$$

We then obtain the spectral radius of $\rho\left(FV^{-1}\right)$, which is defined as the largest eigenvalue of FV^{-1}.

Thus the basic reproduction number for drug sensitive TB, R_{0S} is

$$R_{0S} = \frac{\upsilon(1-\rho_S)\beta_S\pi(\theta+\mu-\mu\sigma)}{\mu(\theta+\mu)(\upsilon+\mu)(\mu+\mu_T+r_2)} + \frac{\rho_S\beta_S\pi(\theta+\mu-\mu\sigma)}{\mu(\theta+\mu)(\mu+\mu_T+r_2)} \tag{20}$$

3.4 Stability of the disease-free equilibrium point with drug sensitive TB

In this section we determine the stability of the disease-free equilibrium points.

3.4.1 Local Stability of the disease-free equilibrium point with drug sensitive TB

The Jacobian of system of differential equations (1), (2), (3), (5) and (7) at disease free is

$$J(E_0)=\begin{bmatrix} -(\theta+\mu) & 0 & 0 & 0 & 0 \\ \theta & -\mu & 0 & -\dfrac{\beta_S\pi(\theta+\mu-\mu\sigma)}{\mu(\theta+\mu)} & 0 \\ 0 & 0 & -(\upsilon+\mu) & \dfrac{(1-\rho_R)\beta_S\pi(\theta+\mu-\mu\sigma)}{\mu(\theta+\mu)} & 0 \\ 0 & 0 & \upsilon & \dfrac{\rho_S\beta_S\pi(\theta+\mu-\mu\sigma)}{\mu(\theta+\mu)}-(\mu+\mu_T+r_2) & 0 \\ 0 & 0 & 0 & r_2(1-r) & -\mu \end{bmatrix} \quad (21)$$

The eigen values of $J(E_0)$ are $\lambda_1=-\mu_1, \lambda_2=-\mu_1$ and $\lambda_3=-(\theta+\mu)$ and the other two are solutions of

$$\begin{vmatrix} -d_1-\lambda & c_1 \\ \upsilon & -d_2-\lambda \end{vmatrix}=0 \quad (22)$$

where

$$d_1=\upsilon+\mu,\ \ d_2=\mu+\mu_T+r_2-\frac{\rho_S\beta_S\pi(\theta+\mu-\mu\sigma)}{\mu(\theta+\mu)} \text{ and } c_1=\frac{(1-\rho_S)\beta_S\pi(\theta+\mu-\mu\sigma)}{\mu(\theta+\mu)}$$

From (22), we have

$$p_2\lambda^2+p_1\lambda+p_0=0 \quad (23)$$

where

$$p_2=1,\ \ p_1=d_1+d_2 \text{ and } \ p_0=d_1d_2-\upsilon c_1=(\upsilon+\mu)(\mu+\mu_T+r_2)(1-R_{0S})$$

We apply Routh-Hurwitz criterion which states that all roots of the polynomial (23) have negative real part iff the coefficients p_i, are positive and the determinant of the matrices $H_i>0$ for $i=0,1,2.$ Therefore, $H_1=p_1=d_1+d_2>0$ iff $d_1>d_2$ and $H_2=\begin{vmatrix} p_1 & 0 \\ 1 & p_0 \end{vmatrix}=p_1p_0=d_1{}^2d_2+d_1d_2{}^2-c_1\upsilon(d_1+d_2)>0$ iff $d_1{}^2d_2+d_1d_2{}^2>c_1\upsilon(d_1+d_2).$ Therefore, all the eigen values of the polynomial (23) have negative real parts, implying that $\lambda_4<0$ and $\lambda_5<0.$ Since all the values of $\lambda_i<0,$ for $i=1,2,3,4,5$ when $R_{0s}<1,$ we conclude that the disease-free equilibrium point is locally asymptotically stable.

3.4.2 Global Stability of the disease-free equilibrium point with drug sensitive TB

Theorem 1:
If $R_{0S} \leq 1$, then the disease-free equilibrium $(E_S, I_S) = (0,0)$ of the system is globally asymptotically stable on Ω.

Proof:
We construct the following Lasalle-Lyapunov function $V(E_S, I_S)$ on the positively invariant compact set Ω. Thus on $\Omega, V(E_S, I_S)$ is continuous and non-negative. We define $V(E_S, I_S) = E_S + (\upsilon + \mu)I_S$. Considering equations (3) and (5) of the system, we have

$$\begin{pmatrix} \dot{E}_S \\ \dot{I}_S \end{pmatrix} = \begin{pmatrix} -(\upsilon+\mu) & \dfrac{(1-\rho_S)\beta_S\pi(\theta+\mu-\mu\sigma)}{\mu(\theta+\mu)} \\ \upsilon & \dfrac{\rho_S\beta_S\pi(\theta+\mu-\mu\sigma)}{\mu(\theta+\mu)} - (\mu+\mu_T+r_2) \end{pmatrix} \begin{pmatrix} E_S \\ I_S \end{pmatrix} \tag{24}$$

Thus, equation (24) can be written as $\dot{I} = A(I)$ where

$$A = \begin{pmatrix} -(\upsilon+\mu) & \dfrac{(1-\rho_S)\beta_S\pi(\theta+\mu-\mu\sigma)}{\mu(\theta+\mu)} \\ \upsilon & \dfrac{\rho_S\beta_S\pi(\theta+\mu-\mu\sigma)}{\mu(\theta+\mu)} - (\mu+\mu_T+r_2) \end{pmatrix}, \quad \text{and } I = \begin{pmatrix} E_S \\ I_S \end{pmatrix}$$

If we define $V^T = [\upsilon, \upsilon+\mu]$, then the derivative along the trajectories given by $V = V^T A(I)$ is

$$V^T A(I) = [\upsilon, \upsilon+\mu] \begin{bmatrix} -(\upsilon+\mu) & \dfrac{(1-\rho_S)\beta_S\pi(\theta+\mu-\mu\sigma)}{\mu(\theta+\mu)} \\ \upsilon & \dfrac{\rho_S\beta_S\pi(\theta+\mu-\mu\sigma)}{\mu(\theta+\mu)} - (\mu+\mu_T+r_2) \end{bmatrix} \tag{25}$$

Simplifying equation (25), we have

$$V^T A(I) = \begin{bmatrix} 0 \\ (\upsilon+\mu)(\mu+\mu_T+r_2)(R_{0S}-1) \end{bmatrix}$$

(26)

which is strictly decreasing when $R_{0S} < 1$. Thus, $V \leq (\upsilon+\mu)(\mu+\mu_T+r_2)(R_{0S}-1)$. We define the set $E = \left\{(E_S, I_S) \in \Omega / V(E_S, \dot{I}_S) = 0\right\}$. The largest invariant set is contained in the set E for which $E_S = 0$ or $I_S = 0$. Thus $\dot{V} < 0$ when $R_{0S} < 1$ and If $I_S = 0$ or $R_{0S} = 1, \dot{V} = 0$. Thus, by Lasalle's invariance principle the disease free equilibrium is globally asymptotically stable on Ω.

3.5 Reproduction number for drug-resistant TB

From equations (3) and (5) we have

$$\frac{dE_R}{dt} = (1-\rho_R)\beta_R I_R S - (\upsilon+\mu)E_R$$

$$\frac{dI_R}{dt} = \rho_R\beta_R I_R S + \upsilon E_R + r_2 r I_S - (\mu+\mu_T+r_1)I_R$$

We evaluated F and V, which are the Jacobian matrices of $\hat{F}$ and $\hat{V}$, respectively at disease free equilibrium E_0 to get

$$F = \begin{bmatrix} 0 & \dfrac{(1-\rho_R)\beta_R\pi(\theta+\mu-\mu\sigma)}{\mu(\theta+\mu)} \\ 0 & \dfrac{\rho_R\beta_R\pi(\theta+\mu-\mu\sigma)}{\mu(\theta+\mu)} \end{bmatrix} \text{ and } V = \begin{bmatrix} (\upsilon+\mu) & 0 \\ -\upsilon & (\mu+\mu_T+r_1) \end{bmatrix}$$

We then obtain the spectral radius of $\rho\left(FV^{-1}\right)$, which is defined as the largest eigenvalue of FV^{-1}. Thus the basic reproduction number for drug resistant TB, R_{0R} is

$$R_{0R} = \frac{\upsilon(1-\rho_R)\beta_R\pi(\theta+\mu-\mu\sigma)}{\mu(\theta+\mu)(\upsilon+\mu)(\mu+\mu_T+r_1)} + \frac{\rho_R\beta_R\pi(\theta+\mu-\mu\sigma)}{\mu(\theta+\mu)(\mu+\mu_T+r_1)} \tag{27}$$

3.6 Stability of the disease-free equilibrium point with drug resistant TB

In this section we determine the stability of the disease-free equilibrium points.

3.6.1 Local Stability of the disease-free equilibrium point with drug resistant TB

The Jacobian of system of differential equations (1), (2), (4), (6) and (8) at disease free is

$$J(E_0) = \begin{bmatrix} -(\theta+\mu) & 0 & 0 & 0 & 0 \\ \theta & -\mu & 0 & -\dfrac{\beta_R\pi(\theta+\mu-\mu\sigma)}{\mu(\theta+\mu)} & 0 \\ 0 & 0 & -(\upsilon+\mu) & \dfrac{(1-\rho_R)\beta_R\pi(\theta+\mu-\mu\sigma)}{\mu(\theta+\mu)} & 0 \\ 0 & 0 & \upsilon & \dfrac{\rho_R\beta_R\pi(\theta+\mu-\mu\sigma)}{\mu(\theta+\mu)} - (\mu+\mu_T+r_1) & 0 \\ 0 & 0 & 0 & r_1 & -\mu \end{bmatrix}$$

The eigen values of $J(E_0)$ are $\lambda_1 = -\mu_1, \lambda_2 = -\mu_1$ and $\lambda_3 = -(\theta+\mu)$ and the other two are solutions of

$$\begin{vmatrix} -(\upsilon+\mu)-\lambda & \dfrac{(1-\rho_R)\beta_R\pi(\theta+\mu-\mu\sigma)}{\mu(\theta+\mu)} \\ \upsilon & -((\mu+\mu_T+r_1)-\dfrac{\rho_R\beta_R\pi(\theta+\mu-\mu\sigma)}{\mu(\theta+\mu)})-\lambda \end{vmatrix} = 0 \quad (28)$$

Now, equation (35) becomes

$$\begin{vmatrix} -D_1-\lambda & C_1 \\ \upsilon & -D_2-\lambda \end{vmatrix} = 0 \quad (29)$$

where

$$D_1 = \upsilon+\mu, D_2 = \mu+\mu_T+r_1-\frac{\rho_R\beta_R\pi(\theta+\mu-\mu\sigma)}{\mu(\theta+\mu)} \text{ and } C_1 = \frac{(1-\rho_R)\beta_R\pi(\theta+\mu-\mu\sigma)}{\mu(\theta+\mu)}$$

From (29), we have

$$a_2\lambda^2 + a_1\lambda + a_0 = 0 \quad (30)$$

where $a_2 = 1, a_1 = d_1 + d_2$ and $a_0 = D_1D_2 - \upsilon C_1 = (\upsilon+\mu)(\mu+\mu_T+r_1)(1-R_{0R})$

We apply Routh-Hurwitz criterion which states that all roots of the polynomial (30) have negative real part iff the coefficients a_i, are positive and the determinant of the matrices $H_i > 0$ for $i = 0,1,2.$ Therefore, $H_1 = a_1 = D_1 + D_2 > 0$ iff $D_1 > D_2$ and

$$H_2 = \begin{vmatrix} a_1 & 0 \\ 1 & a_0 \end{vmatrix} = a_1a_0 = D_1^2D_2 + D_1D_2^2 - C_1\upsilon(D_1+D_2) > 0 \text{ iff } D_1^2D_2 + D_1D_2^2 > C_1\upsilon(D_1+D_2).$$

Therefore, all the eigen values of the polynomial (30) have negative real parts, implying that $\lambda_4 < 0$ and $\lambda_5 < 0.$ Since all the values of $\lambda_i < 0,$ for $i = 1,2,3,4,5$ when $R_{0R} < 1,$ we conclude that the disease-free equilibrium point is locally asymptotically stable.

3.6.2 Global stability of the DFE with drug-resistant TB

Theorem 2:
If $R_{0R} \leq 1,$ then the disease-free equilibrium $(E_R, I_R) = (0,0)$ of the system is globally asymptotically stable on $\Omega.$

Proof:
We construct the following Lasalle-Lyapunov function $V(E_R, I_R)$ on the positively invariant compact set $\Omega.$ Thus on $\Omega,$ $V(E_R, I_R)$ is continuous and non-negative. We define $V(E_R, I_R) = E_R + (\upsilon+\mu)I_R.$ Considering equations (4) and (6) of the system, we have

$$\begin{pmatrix} \dot{E}_R \\ \dot{I}_R \end{pmatrix} = \begin{pmatrix} -(\upsilon+\mu) & \dfrac{(1-\rho_R)\beta_R\pi(\theta+\mu-\mu\sigma)}{\mu(\theta+\mu)} \\ \upsilon & \dfrac{\rho_R\beta_R\pi(\theta+\mu-\mu\sigma)}{\mu(\theta+\mu)} - (\mu+\mu_T+r_1) \end{pmatrix} \begin{pmatrix} E_R \\ I_R \end{pmatrix} \tag{31}$$

Thus, equation (31) can be written as $\dot{I} = A(I),$ where

$$A = \begin{pmatrix} -(\upsilon+\mu) & \dfrac{(1-\rho_R)\beta_R\pi(\theta+\mu-\mu\sigma)}{\mu(\theta+\mu)} \\ \upsilon & \dfrac{\rho_R\beta_R\pi(\theta+\mu-\mu\sigma)}{\mu(\theta+\mu)} - (\mu+\mu_T+r_1) \end{pmatrix} \text{ and } I = \begin{pmatrix} E_R \\ I_R \end{pmatrix}$$

If we define $V^T = [\upsilon, \upsilon+\mu],$ then the derivative along the trajectories given by $V = V^T A(I)$ is

$$V^T A(I) = [\upsilon, \upsilon+\mu] \begin{bmatrix} -(\upsilon+\mu) & \dfrac{(1-\rho_R)\beta_R\pi(\theta+\mu-\mu\sigma)}{\mu(\theta+\mu)} \\ \upsilon & \dfrac{\rho_R\beta_R\pi(\theta+\mu-\mu\sigma)}{\mu(\theta+\mu)} - (\mu+\mu_T+r_1) \end{bmatrix} \tag{32}$$

Simplifying equation (32), we have

$$V^T A(I) = \begin{bmatrix} 0 \\ (\upsilon+\mu)(\mu+\mu_T+r_1)(R_{0R}-1) \end{bmatrix}$$

(33)

which is strictly decreasing when $R_{0R} < 1.$ Thus, $V \le (\upsilon+\mu)(\mu+\mu_T+r_1)(R_{0R}-1).$ We define the set $E = \left\{(E_R, I_R) \in \Omega / V(E_R, \dot{I}_R) = 0\right\}.$ The largest invariant set is contained in the set E for which

$E_R = 0$ or $I_R = 0.$ Thus $\dot{V} < 0$ when $R_{0R} < 1.$ If $I_R = 0$ or $R_{0R} = 1,$ $\dot{V} = 0.$ Thus, by Lasalle's invariance principle the disease free equilibrium is globally asymptotically stable on $\Omega.$

3.7 Reproduction number for both drug sensitive and drug-resistant TB

From equations (3), (4), (5) and (6) we have

$$\frac{dE_S}{dt} = (1-\rho_S)\beta_S I_S S - (\upsilon+\mu)E_S$$

$$\frac{dE_R}{dt} = (1-\rho_R)\beta_R I_R S - (\upsilon+\mu)E_R$$

$$\frac{dI_S}{dt} = \rho_S\beta_S I_S S + \upsilon E_S - (\mu+\mu_T+r_2)I_S$$

$$\frac{dI_R}{dt} = \rho_R \beta_R I_R S + \upsilon E_R + r_2 r I_S - (\mu + \mu_T + r_1) I_R$$

We evaluated F and V, which are the Jacobian matrices of $\hat{F}$ and $\hat{V}$, respectively at disease free equilibrium E_0 to get

$$F = \begin{bmatrix} 0 & 0 & \frac{(1-\rho_S)\beta_S \pi(\theta+\mu-\mu\sigma)}{\mu(\theta+\mu)} & 0 \\ 0 & 0 & 0 & \frac{(1-\rho_R)\beta_R \pi(\theta+\mu-\mu\sigma)}{\mu(\theta+\mu)} \\ 0 & 0 & \frac{\rho_S \beta_S \pi(\theta+\mu-\mu\sigma)}{\mu(\theta+\mu)} & 0 \\ 0 & 0 & 0 & \frac{\rho_R \beta_R \pi(\theta+\mu-\mu\sigma)}{\mu(\theta+\mu)} \end{bmatrix} \text{ and}$$

$$V = \begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_1 & 0 & 0 \\ -\upsilon & 0 & A_2 & 0 \\ 0 & -\upsilon & -r_2 r & A_3 \end{bmatrix}$$

where

$$A_1 = (\upsilon + \mu)$$

$$A_2 = (\mu + \mu_T + r_2)$$

$$A_3 = (\mu + \mu_T + r_1)$$

We then obtain the spectral radius of $\rho\left(FV^{-1}\right)$, which is defined as the largest eigenvalue of FV^{-1}. The reproduction number for both drug sensitive TB and drug resistant TB is given as $R_0 = \max\{R_{0S}, R_{0R}\}$, where R_{0S} and R_{0R} are reproduction numbers for drug sensitive TB strain and drug resistant TB strain respectively.

3.8 Local stability of the DFE with both drug sensitive TB and drug- resistant TB

The Jacobian of system of differential equations (1) - (8) at disease free is

$$J(E_0)=\begin{bmatrix} -a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \theta & -\mu & 0 & 0 & b_1 & b_4 & 0 & 0 \\ 0 & 0 & -a_2 & 0 & b_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_3 & 0 & b_5 & 0 & 0 \\ 0 & 0 & \upsilon & 0 & -a_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & \upsilon & r_2 r & -a_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_3 & 0 & -\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & r_1 & 0 & -\mu \end{bmatrix}$$

Where $a_1=(\theta+\mu), a_2=a_3=(\upsilon+\mu), a_4=(\mu+\mu_T+r_2)$ and $a_5=(\mu+\mu_T+r_1),$

The eigen values of $J(E_0)$ are $\lambda_1=-\mu_1, \lambda_2=-\mu_1, \lambda_3=-\mu_1$ and $\lambda_4=-(\theta+\mu)$ and the other four are solutions of the characteristics equation

$$m_4\lambda^4+m_3\lambda^3+m_2\lambda^2+m_1\lambda+m_0=0 \quad (34)$$

where

$$m_4=1$$

$$m_3=a_2+a_3+a_4+a_5$$

$$m_2=\left(a_2a_4+a_3a_4+a_2a_5+a_4a_5+a_2a_3+a_3a_5-\upsilon\left(b_2+b_5\right)\right)$$

$$m_1=\left(a_2a_3a_4+a_2a_3a_5+a_2a_4a_5+a_3a_4a_5-\upsilon\left(a_2b_5+a_3b_2+a_4b_5+a_5b_2\right)\right)$$

$$m_0=a_2a_3a_4a_5+\upsilon^2b_2b_5-\upsilon(a_2a_4b_5+a_3a_5b_2)$$

We apply Routh-Hurwitz criterion which states that all roots of the polynomial (34) have negative real part iff the coefficients $m_i,$ are positive and the determinant of the matrices $H_i>0$ for $i=0,1,2,3,4$ therefore,

$$H_1=m_3=a_2+a_3+a_4+a_5>0$$

$$H_2=\begin{vmatrix} m_3 & m_1 \\ 1 & m_2 \end{vmatrix}=m_3m_2-m_1>0, \text{ iff } \ m_3m_2>m_1$$

$$H_3 = \begin{vmatrix} m_3 & m_1 & 0 \\ 1 & m_2 & m_0 \\ 0 & m_3 & m_1 \end{vmatrix} = m_1 m_2 m_3 - (m_0 m_3^{\,2} + m_1^{\,2}) > 0 \text{ iff } m_1 m_2 m_3 > m_0 m_3^{\,2} + m_1^{\,2}$$

$$H_4 = \begin{vmatrix} m_3 & m_1 & 0 & 0 \\ 1 & m_2 & m_0 & 0 \\ 0 & m_3 & m_1 & 0 \\ 0 & 1 & m_2 & m_0 \end{vmatrix} = m_0 m_1 m_2 m_3 - (m_0 m_1^{\,2} + m_0^{\,2} m_3^{\,2}) > 0 \text{ iff } m_0 m_1 m_2 m_3 > (m_0 m_1^{\,2} + m_0^{\,2} m_3^{\,2})$$

Therefore, all the eigen values of the polynomial (34) have negative real parts, implying that $\lambda_5 < 0, \lambda_6 < 0, \lambda_7 < 0$ and $\lambda_8 < 0.$ Since all the values of $\lambda_i < 0,$ for $i = 1,2,3,4,5,6,7,8$ when $R_0 < 1,$ we conclude that the disease-free equilibrium point is locally asymptotically stable.

3.9 Global stability of the DFE with both drug resistant TB and drug resistant TB

From equations (3), (4), (5) and (6) we have

$$\frac{dE_S}{dt} = (1-\rho_S)\beta_S I_S S - (\upsilon + \mu) E_S$$

$$\frac{dI_S}{dt} = \rho_S \beta_S I_S S + \upsilon E_S - (\mu + \mu_T + r_2) I_S$$

$$\frac{dE_R}{dt} = (1-\rho_R)\beta_R I_R S - (\upsilon + \mu) E_R$$

$$\frac{dI_R}{dt} = \rho_R \beta_R I_R S + \upsilon E_R + r_2 r I_S - (\mu + \mu_T + r_2) I_R$$

$$V = \upsilon E_S + (\upsilon + \mu) I_S + \upsilon E_R + (\upsilon + \mu) I_R \tag{35}$$

Differentiating equation (35) we get

$$\dot{V} = \upsilon \dot{E_S} + (\upsilon + \mu)\dot{I_S} + \upsilon \dot{E_R} + \upsilon + \mu \dot{I_R} \tag{36}$$

Substituting equations (3), (4), (5) and (6) and careful simplification gives

$$\dot{V} \le (\upsilon + \mu)(\mu + \mu_T + r_2)[R_{0S} - 1] I_S + (\upsilon + \mu)(\mu + \mu_T + r_1)[R_{0R} - 1] I_R \tag{37}$$

Thus, from equation (37), $R_0 \le 1$ implies that $\dot{V} \le 0.$ By Lasalle's invariance principle, the largest invariant set $\Omega_E,$ contained $\left\{(E_S, E_R, I_S, I_R) \in \Omega_E, \dot{V} = 0\right\}$ is reduced to the disease-free equilibrium $E_0.$ This proves the global asymptotic stability of the disease-free equation on $\Omega_E.$

4 Numerical Simulation

We perform some numerical experiments using ode45 function from MATLAB R2015a to study the behaviour of the system and effects of passive immunity, drug sensitive and drug resistance on the

transmission dynamics of tuberculosis. The initial condition for each plot and parameter values are presented in table 2.

Variable/Parameter	Values	Source(s)
β_S	0.0290	[12]
β_R	0.0290	[12]
π	200	[12]
μ	0.02	[3]
μ_T	0.3	[3]
υ	0.0013	[3]
ρ_S	0.1	[3]
ρ_R	0.1	[3]
r_1	0.2	[3]
r_2	0.3	[3]
θ	0.7	[3]
r	0.9	Assumed
σ	0.10	[4]
$M(0)$	950	Assumed
$S(0)$	3800	[4]
$E_S(0)$	1800	[4]
$E_R(0)$	100	[4]
$I_S(0)$	200	[4]
$I_R(0)$	50	[4]
$R_S(0)$	30	Assumed
$R_R(0)$	20	Assumed

Table 2: Variables and parameters values used for computational results

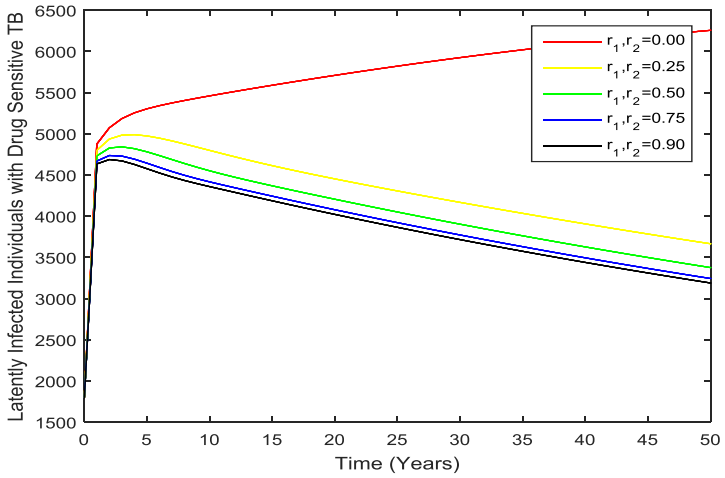

Figure 2. The effect of treatment efficacy of both DS-TB and DR-TB on latently infected individuals with DS-TB over time. The following parameter values were used: $r_1, r_2 = (0.00 - 0.90)$, $\beta_S = 0.029, \beta_R = 0.029$ and $r = 0.10$. All other parameter values are stated in table 2.

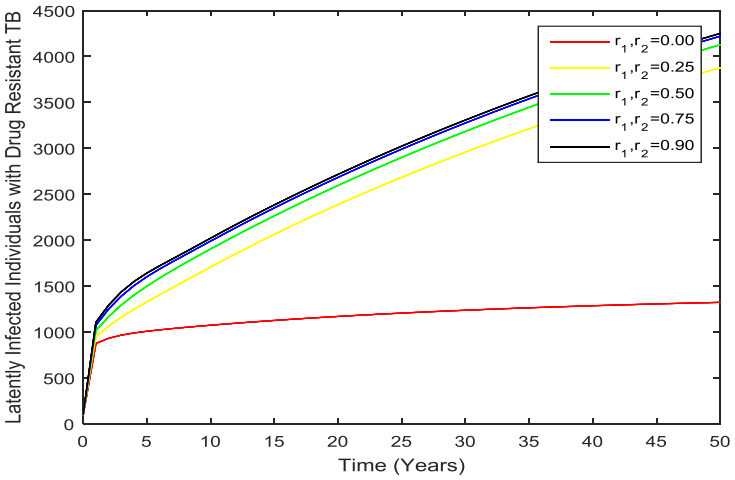

Figure 3. The effect of treatment efficacy of both DS-TB and DR-TB on latently infected individuals with DR-TB over time. The following parameter values were used: $r_1, r_2 = (0.00 - 0.90)$, $\beta_S = 0.029, \beta_R = 0.029$ and $r = 0.10$. All other parameter values are stated in table 2

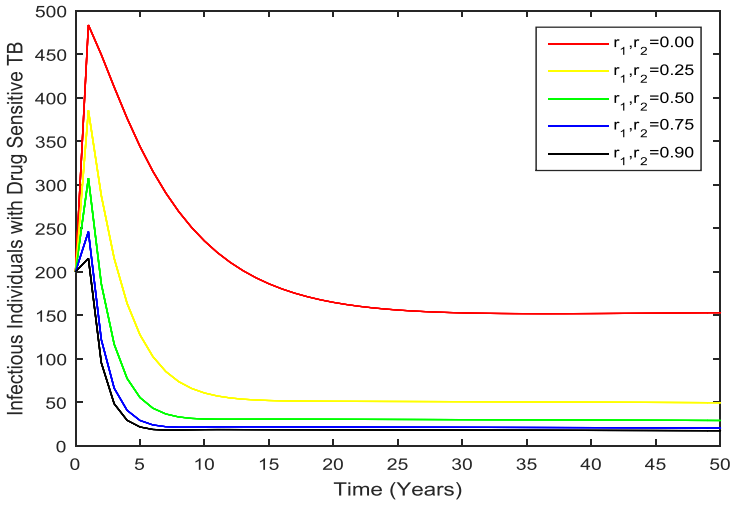

Figure 4. The effect of treatment efficacy of both DS-TB and DR-TB on infectious individuals with DS-TB over time. The following parameter values were used: $r_1, r_2 = (0.00 - 0.90)$, $\beta_S = 0.029, \beta_R = 0.029$ and $r = 0.10$. All other parameter values are stated in table 2

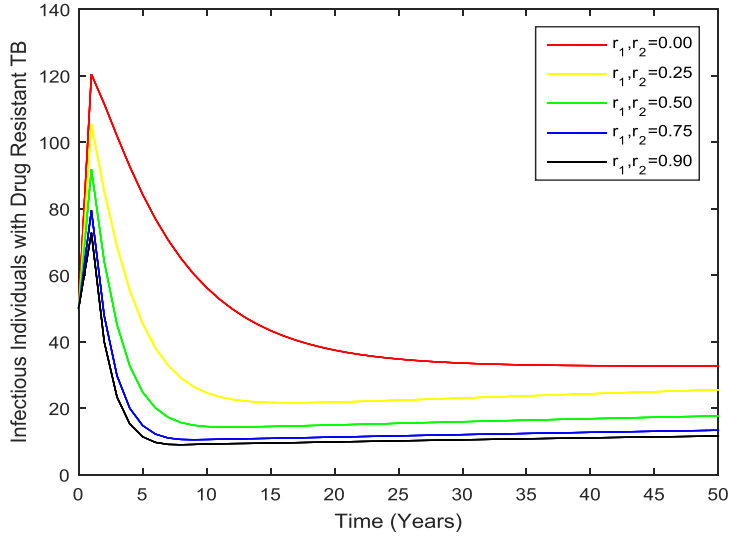

Figure 5. The effect of treatment efficacy of both DS-TB and DR-TB on infectious individuals with DR-TB over time. The following parameter values were used: $r_1, r_2 = (0.00 - 0.90)$, $\beta_S = 0.029, \beta_R = 0.029$ and $r = 0.10$. All other parameter values are stated in table 2.

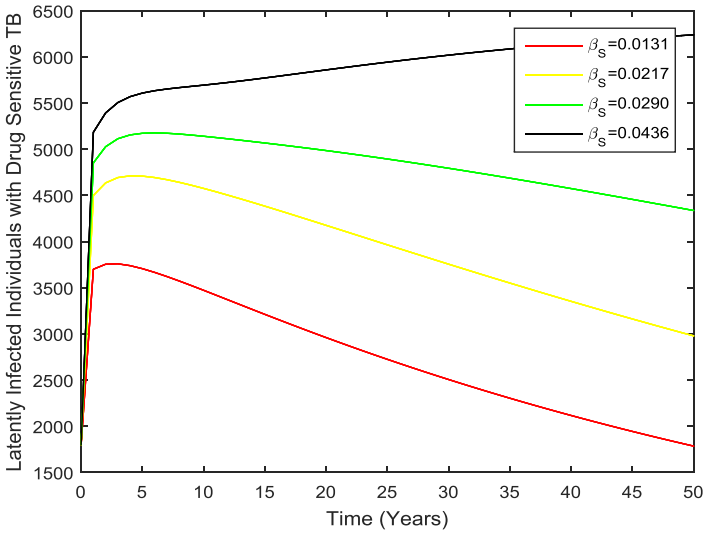

Figure 6. The effect of transmission rate of DS-TB on latently infected individuals with DS-TB over time. The following parameter values were used: $\beta_R = 0.029, r = 0.10$ and $\beta_S = (0.0131 - 0.0436)$. All other parameter values are stated in table 2.

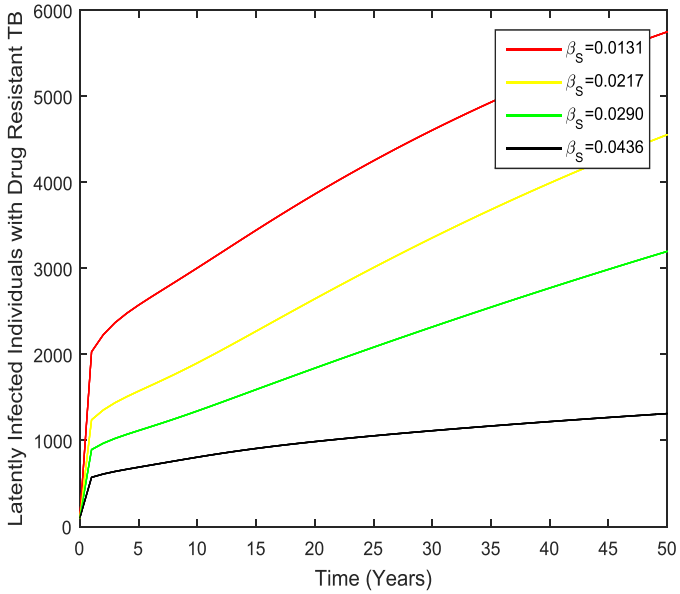

Figure 7. The effect of transmission rate of DS-TB on latently infected individuals with DR-TB over time. The following parameter values were used: $\beta_R = 0.029, r = 0.10$ and $\beta_S = (0.0131 - 0.0436)$. All other parameter values are stated in table 2.

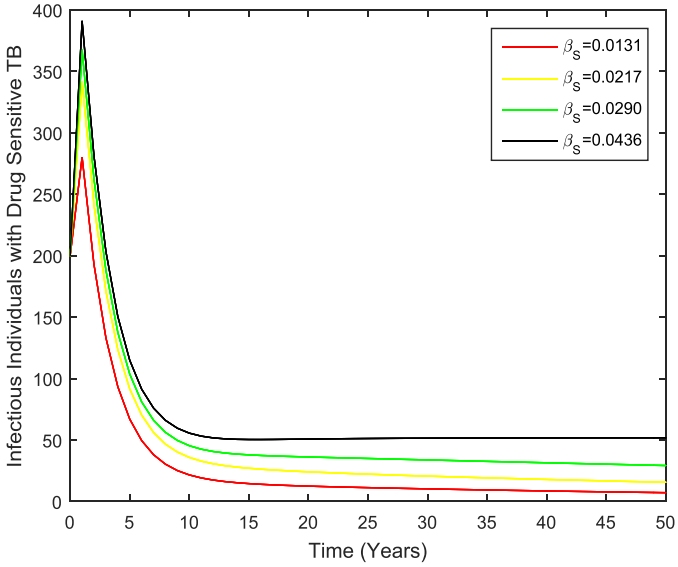

Figure 8. The effect of transmission rate of DS-TB on infectious individuals with DS-TB over time. The following parameter values were used: $\beta_R = 0.029, r = 0.10$ and $\beta_S = (0.0131 - 0.0436)$. All other parameter values are stated in table 2

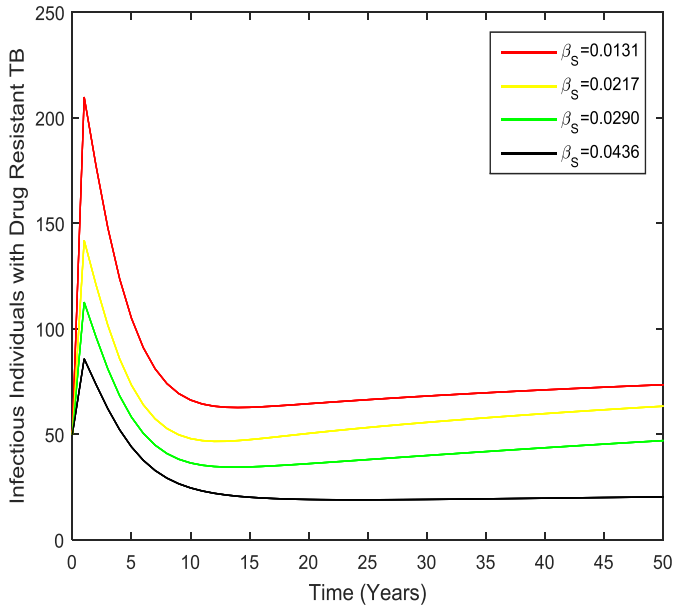

Figure 9. The effect of transmission rate of DS-TB on infectious individuals with DR-TB over time. The following parameter values were used: $\beta_R = 0.029, r = 0.10$ and $\beta_S = (0.0131 - 0.0436)$. All other parameter values are stated in table 2

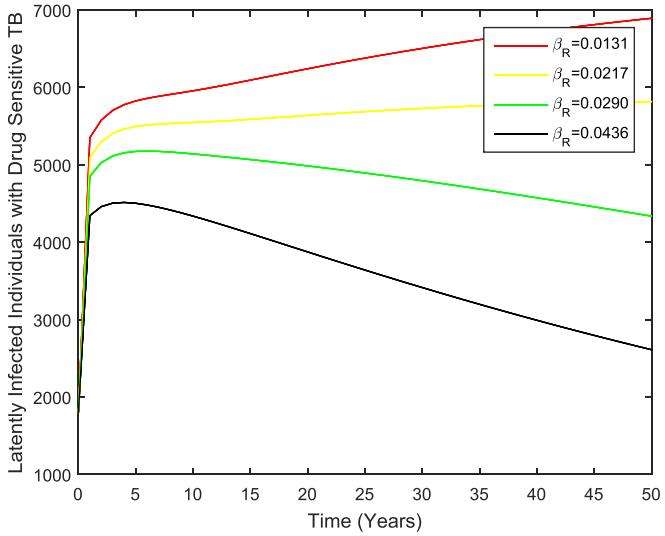

Figure 10. The effect of transmission rate of DR-TB on latently infected individuals with DS-TB over time. The following parameter values were used: $\beta_S = 0.029, r = 0.10$ and $\beta_R = (0.0131 - 0.0436)$. All other parameter values are stated in table 2.

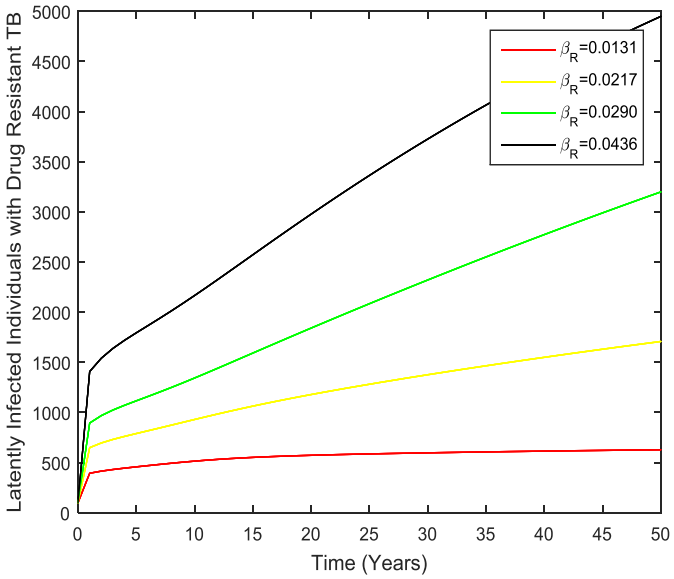

Figure 11. The effect of transmission rate of DR-TB on latently infected individuals with DR-TB over time. The following parameter values were used: $\beta_S = 0.029, r = 0.10$ and $\beta_R = (0.0131 - 0.0436)$. All other parameter values are stated in table 2.

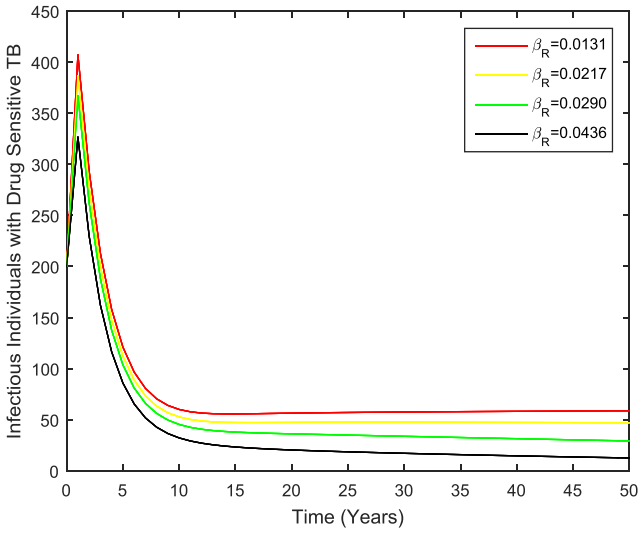

Figure 12. The effect of transmission rate of DR-TB on infectious individuals with DS-TB over time. The following parameter values were used: $\beta_S = 0.029, r = 0.10$ and $\beta_R = (0.0131 - 0.0436)$. All other parameter values are stated in table 2.

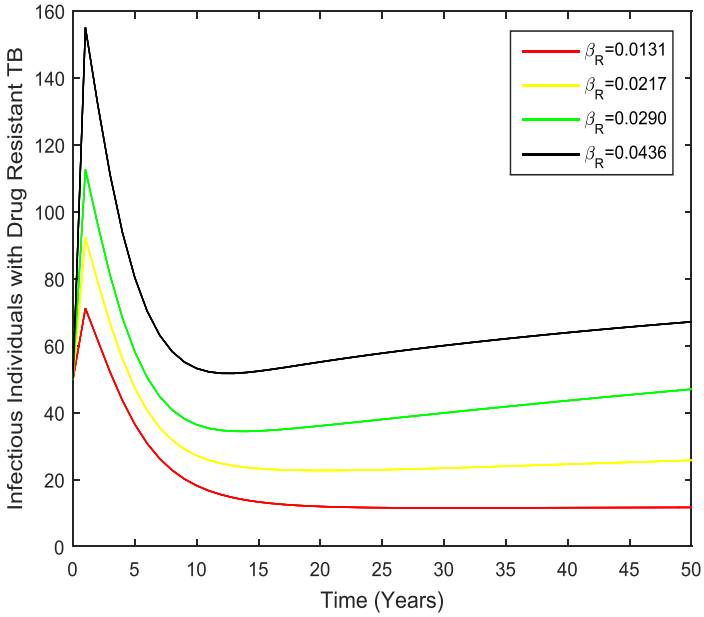

Figure 13. The effect of transmission rate of DR-TB on infectious individuals with DR-TB over time. The following parameter values were used: $\beta_S = 0.029, r = 0.10$ and $\beta_R = (0.0131 - 0.0436)$. All other parameter values are stated in table 2.

5 Discussion of Results

In Figure (2) it was observed that there is a significant impact in the latently infected individuals with DS-TB class. This implies that as the treatment efficacy of both DS-TB and DR-TB rapidly increases, individuals with latently infected with DS-TB decreases due to proper treatment. Figure (3) has shown no effect in the latently infected individuals with DR-TB class. This implies that as the treatment efficacy of both DS-TB and DR-TB increases, then individuals with latently infected with DR-TB decreases because more individuals are getting infected due to treatment failure. In this case, identifying and curing latently infected individuals with DS-TB becomes more important in the reduction of new cases of DR-TB.

In Figure (4), observed that there a significant impact in the infectious individuals with DS-TB class. This implies that as the treatment efficacy of both DS-TB and DR-TB increases, then infectious individuals with DS-TB decreases and stable at a minimal point due to treatment efficacy and as a result of that DS-TB is curable. Similarly, Figures (5) follows the same pattern as that of figure (4). Therefore, DR-TB is also curable. In Figure (6), observed that as the transmission rate of DS-TB increases and transmission rate of DR-TB is constant, then latently infected individuals with DS-TB class increases due to the fact that more people are coming from susceptible class. Figure (7) has no effect in the latently infected individuals with DR-TB class. This implies that as the transmission rate of DS-TB increases and transmission rate of DR-TB is constant, then latently infected individuals with DR-TB class decreases due to the fact that many individuals are moving into class of DS-TB. There is significant impact in the infectious individuals with DS-TB class as shown in figure (8). This implies that as the transmission rate of DS-TB increases and transmission rate of DR-TB is constant, then infectious individuals with DS-TB class increases exponentially due to the fact that more people are coming from susceptible and latently infected individuals with DS-TB classes. Figure (9) shows no effect in the infectious individuals with DR-TB due to the fact that infectious individuals with DR-TB decreases and stable at a minimal point while Figure (10) has shown no effect in the latently infected individuals with DS-TB class. This implies that as the transmission rate of DR-TB increases and transmission rate of DS-TB is constant, then latently infected individuals with DS-TB class decreases since more individuals are moving to latently infected individuals with DR-TB than that of DS-TB. Figure (11) shows a significant impact in the infected individuals with DR-TB class. This implies that as the transmission rate of DR-TB increases and transmission rate of DS-TB is constant, then latently infected individuals with DR-TB class increases exponentially due to the fact that more people are coming from susceptible class and as a result of treatment failure than from primary infections. In this case, curing latently infected individuals with DR-TB becomes more important in the reduction of new cases of DR-TB.

From figure (12), we observed that there is no effect on the infectious individuals with DR-TB class. This implies that as the transmission rate of DR-TB increases and transmission rate of DR-TB is constant, then infectious individuals with DS-TB decreases naturally and stable at a particular point. Figure (13) shows a significant impact in the infectious individuals with DR-TB class. It implies that as the transmission rate of DS-TB increases and transmission rate of DR-TB is constant, then infectious individuals with DR-TB class increases due to the fact that more people are coming from susceptible, latently infected individuals with DR-TB and infectious individuals with DS-TB classes.

Conclusion

This study presents a simple yet more realistic deterministic model for the transmission dynamics of tuberculosis. In contrast to many tuberculosis models in literature, we incorporate passive immunity, drug sensitive class and drug resistant to the first line of treatment for tuberculosis. Analytical study was carried out using linearized stability and the results shows that the disease free equilibrium (DFE) points are locally asymptotically stable (LAS) whenever $R_0 < 1$ and global asymptotically stable (GAS) whenever $R_0 \leq 1$. The result of the numerical experiment carried out indicates that behavioural change significantly improves the rate at which tuberculosis is eradicated from the society.

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