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The stability of an inhomogeneous rational integrator for the solution in ordinary differential equations

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Abstract

This research work is concerned with the determination of solution to different classes of problems in Ordinary Differential Equations (ODEs). We derived an inhomogeneous rational integrators for the solution of ordinary differential equations. The convergence and consistency and scheme in the interpolant of order m = 4, was established. The stability function and the analysis of the method was carried out with the use of MAPLE-18 and MATLAB software. We compared our new integrator with some recent existing methods and discovered that ours compete favorably well with a very high rate of convergence..Our result shows that the new integrator is stable analytically and computationally.

Keywords and Phrases: rational integrator; stiff; singular and oscillatory problems; stability function and region of absolute stability (RAS)

1 Introduction

In physical sciences, mathematical models are developed to help in the understanding of physical phenomena. These models often yield equations that contain some derivatives of an unknown function of one or several variables. Such equation are called differential equation, [13]. Differential equations do not only arise in the physical sciences but also in diverse fields as economics, medicine, engineering, psychology, operation research and even in areas such as biological simulations and anthropology, population models, electrical, network, nuclear reactors, mechanical oscillations, chemical kinetics, engineering work, process control [5].

According to [8], one of the main directions to construct methods with higher order and extensively stability region, is using higher derivatives of the solutions and some methods that have been introduced that have good properties, especially for stiff problems in [2],

$$y' = f(x, y), \quad y(x_0) = y_0, \;\; a \le x \le b, \tag{1}$$

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where $f(x, y)$ must satisfy a Lipschitz condition with respect to y and is defined and continuous in a region D ⊂ [a,b] that are either Stiff or Singular.

Lipschitz constant according to [2], from (1) above, $f(x, y)$ is said to satisfy a Lipschitz condition in y, over the region D, if there exist a constant L such that

$$||f(x, y_1) - f(x, y_2)|| \leq L||y_1 - y_2|| \tag{2}$$

where L is called the Lipschitz constant and $f(x, y)$ is said to be Lipschitzian by virtue of the relation.

Definition 1: Stiffness [6]

The system (1) is said to be stiff over the interval [a,b] if for every $x\epsilon[a, b]$, the eigenvalues $\{\lambda,(x), s = 1,\ldots, m\}$ of the Jacobian matrix

$$J = \frac{\partial f}{\partial y} \tag{3}$$

Satisfy the following conditions:

Re $\lambda_s(x) < 0, s = 1, \ldots, m$,

where Re $\lambda_s(x)$ represents the real component of the complex number $\lambda_s(x)$

Stiffness Ratio = $\frac{\lambda_r(x)}{\lambda_s(x)} >> 1$, r,s = 1,…,m.

For the linear initial value problem

$$y' = Ay + g, \qquad y(x_0) = y_0$$

$$\frac{\partial f}{\partial y} = \frac{\partial f'}{\partial y} = A$$

is m x m matrix and $g\epsilon R^m$.

Definition 2: [7]: A numerical method is said to be A-stable if its region of absolute stability contains the whole of the left-hand half-plane $Re\ h\lambda < 0$.

However, A-stable is a severe requirement to ask of a numerical method for stiffness to be satisfied.

Definition 3: [11]: A numerical method is said to be stiffly-stable if

the region of absolute stability contains $\Re_1\ and\ \Re_1$ and

(ii) it is accurate for all $h \in \Re_2$ when applied to the test equation $y' = \lambda y, \lambda$ a complex constant with $Re\ h\lambda < 0$, where

$\Re_1 = \{[h\lambda] Re\ h\lambda < -a\}$,

$\Re_2 = \{[h\lambda] - a \leq Re\ h\lambda \leq b, -c \leq Imh\lambda \leq c\}$,

and $a, b\ and\ c$ are positive constants.

Definition 4: [1]: A one-step numerical method is said to be L-stable if it is A –stable and, in addition, when applied to the scalar test equation $y' = \lambda y,\ \lambda$ is a complex constant with $Re\ h\lambda < 0$, it yields $y_{n+1} = Re\ (h\lambda)y_n$, where $|Re\ (h\lambda)| \to 0$ as $Re\ (h\lambda) \to \infty$.

2 Derivation of the method

The rational interpolant U : R → R is defined by

$$y_{n+1} = r + \frac{\sum_{i=0}^{m-1} p_i x_{n+1}^i}{1 + \sum_{i=1}^{m} q_i x_{n+1}^i} \tag{4}$$

where p_i *and* q_i are called the integrator parameters, and r is an inhomogeneous constant.

At m = 4, from (4) we have

$$y_{n+1} = r + \frac{\sum_{i=0}^{3} p_i x_{n+1}^i}{1 + \sum_{i=1}^{4} q_i x_{n+1}^i} \tag{5}$$

where $p_{0,}\ p_{1,}\ p_{2,}\ p_{3,}$ *and* $q_{1,}\ q_{2,}\ q_{3,}\ q_{4,}$ are called the integrator parameters and r is a constant. Also

$$\begin{aligned} y_{n+1} &= r + [p_0 + p_1 x + p_2 x^2 + p_3 x^3][1 + q_1 x + q_2 x^2 + q_3 x^3 + q_4 x^4]^{-1} \\ &= r + [p_0 + p_1 x + p_2 x^2 + p_3 x^3]\{\textstyle\sum_{i=0}^{\infty} (-1)^i [\, q_1 x + q_2 x^2 + q_3 x^3 + q_4 x^4]^i\}. \end{aligned}$$

By the use of binomial expansion for rational functions, we have,

$$\begin{aligned} y_{n+1} &= r + [p_0 + p_1 x + p_2 x^2 + p_3 x^3][1 - (q_1 x + q_2 x^2 + q_3 x^3 + q_4 x^4) + (q_1 x + q_2 x^2 \\ &\quad + q_3 x^3 \\ &\quad + q_4 x^4)^2 - (q_1 x + q_2 x^2 + q_3 x^3 + q_4 x^4)^3 + (q_1 x + q_2 x^2 + q_3 x^3 + q_4 x^4)^4 - \cdots] \end{aligned}$$

$$y_n = r + p_0$$

$$r = y_n - p_0 \tag{6}$$

$$p_1 = \frac{h y_n^{(1)}}{1!\, x_{n+1}} + p_0 q_1$$

$$p_1 x_{n+1} = h y_n^{(1)} + p_0 q_1 x_{n+1} \tag{7}$$

$$p_2 = \frac{h^2 y_n^{(2)}}{2!\, x_{n+1}^2} + \frac{h y_n^{(1)}}{1!\, x_{n+1}} + p_0 q_2$$

$$p_2 x_{n+1}^2 = \frac{h^2 y_n^{(2)}}{2!} + h y_n^{(1)} q_1 x_{n+1} + p_0 q_2 x_{n+1}^2 \tag{8}$$

$$p_3 = \frac{h^3 y_n^{(3)}}{3! x_{n+1}^3} + \frac{h^2 y_n^{(2)}}{2! x_{n+1}^2} q_1 + \frac{h^1 y_n^{(1)}}{1! x_{n+1}} q_2 + p_0 q_3$$

$$p_3x_{n+1}^3=\frac{h^3y_n^{(3)}}{3!}+\frac{h^2y_n^{(2)}q_1x_{n+1}}{2!}+hy_n^{(1)}q_2x_{n+1}^2+p_0q_3x_{n+1}^3 \tag{9}$$

$$-p_0q_4=\frac{h^4y_n^{(4)}}{4!\,x_{n+1}^4}+\frac{h^3y_n^{(3)}}{3!\,x_{n+1}^3}q_1+\frac{h^2y_n^{(2)}}{2!\,x_{n+1}^2}q_2+\frac{h^1y_n^{(1)}}{1!\,x_{n+1}}q_3$$

$$p_0=\frac{-1}{q_4}\left[\frac{h^4y_n^{(4)}}{4!}+\frac{h^3y_n^{(3)}q_1x_{n+1}^1}{3!}+\frac{h^2y_n^{(2)}q_2x_{n+1}^2}{2!}+hy_n^{(1)}q_3x_{n+1}^3\right] \tag{10}$$

$$r=y_n+\frac{-1}{q_4}\left[\frac{h^4y_n^{(4)}}{4!}+\frac{h^3y_n^{(3)}q_1x_{n+1}^1}{3!}+\frac{h^2y_n^{(2)}q_2x_{n+1}^2}{2!}+hy_n^{(1)}q_3x_{n+1}^3\right]. \tag{11}$$

Now from equ. (5), the rational integrator is expanded to give

$$y_{n+1}=\frac{p_0+p_1x_{n+1}+p_2x_{n+1}^2+p_3x_{n+1}^3+r+rq_1x_{n+1}+rq_2x_{n+1}^2+rq_3x_{n+1}^3+rq_4x_{n+1}^4}{1+q_1x_{n+1}+q_2x_{n+1}^2+q_3x_{n+1}^3+q_4x_{n+1}^4}. \tag{12}$$

Then

$$p_0+r=r+p_0=y_n \tag{13}$$

$$p_1x_{n+1}+rq_1x_{n+1}=hy_n^{(1)}-\frac{q_1x_{n+1}}{q_4x_{n+1}^4}\left[\frac{h^4y_n^{(4)}}{4!}+\frac{h^3y_n^{(3)}q_1x_{n+1}^1}{3!}+\frac{h^2y_n^{(2)}q_2x_{n+1}^2}{2!}+hy_n^{(1)}q_3x_{n+1}^3\right]$$
$$+q_1x_{n+1}y_n+\frac{q_1x_{n+1}}{q_4x_{n+1}^4}\left[\frac{h^4y_n^{(4)}}{4!}+\frac{h^3y_n^{(3)}q_1x_{n+1}^1}{3!}+\frac{h^2y_n^{(2)}q_2x_{n+1}^2}{2!}+hy_n^{(1)}q_3x_{n+1}^3\right]$$

$$p_1x_{n+1}+rq_1x_{n+1}=hy_n^{(1)}+q_1x_{n+1}y_n \tag{14}$$

$$p_2x_{n+1}^2+rq_2x_{n+1}^2=\frac{h^2y_n^{(2)}}{2!}+hy_n^{(1)}q_1x_{n+1}-$$
$$\frac{q_2x_{n+1}^2}{q_4x_{n+1}^4}\left[\frac{h^4y_n^{(4)}}{4!}+\frac{h^3y_n^{(3)}q_1x_{n+1}^1}{3!}+\frac{h^2y_n^{(2)}q_2x_{n+1}^2}{2!}+hy_n^{(1)}q_3x_{n+1}^3\right]+q_2x_{n+1}^2y_n$$
$$+\frac{q_2x_{n+1}^2}{q_4x_{n+1}^4}\left[\frac{h^4y_n^{(4)}}{4!}+\frac{h^3y_n^{(3)}q_1x_{n+1}^1}{3!}+\frac{h^2y_n^{(2)}q_2x_{n+1}^2}{2!}+hy_n^{(1)}q_3x_{n+1}^3\right]$$

$$p_2x_{n+1}^2+rq_2x_{n+1}^2=\frac{h^2y_n^{(2)}}{2!}+hy_n^{(1)}q_1x_{n+1}+q_2x_{n+1}^2y_n \tag{15}$$

$$p_3x_{n+1}^3+rq_3x_{n+1}^3=\frac{h^3y_n^{(3)}}{3!}+\frac{h^2y_n^{(2)}q_1x_{n+1}}{2!}+hy_n^{(1)}q_2x_{n+1}^2-$$

$$\frac{q_3x_{n+1}^3}{q_4x_{n+1}^4}\left[\frac{h^4y_n^{(4)}}{4!}+\frac{h^3y_n^{(3)}q_1x_{n+1}^1}{3!}+\frac{h^2y_n^{(2)}q_2x_{n+1}^2}{2!}+hy_n^{(1)}q_3x_{n+1}^3\right]+\ q_3x_{n+1}^3y_n$$
$$+\frac{q_3x_{n+1}^3}{q_4x_{n+1}^4}\left[\frac{h^4y_n^{(4)}}{4!}+\frac{h^3y_n^{(3)}q_1x_{n+1}^1}{3!}+\frac{h^2y_n^{(2)}q_2x_{n+1}^2}{2!}+hy_n^{(1)}q_3x_{n+1}^3\right]$$

$$p_3x_{n+1}^3+rq_3x_{n+1}^3=\frac{h^3y_n^{(3)}}{3!}+\frac{h^2y_n^{(2)}q_1x_{n+1}}{2!}+hy_n^{(1)}q_2x_{n+1}^2+q_3x_{n+1}^3y_n \tag{16}$$

$$rq_4x_{n+1}^4=q_4x_{n+1}^4y_n+\frac{q_4x_{n+1}^4}{q_4x_{n+1}^4}\left[\frac{h^4y_n^{(4)}}{4!}+\frac{h^3y_n^{(3)}q_1x_{n+1}^1}{3!}+\frac{h^2y_n^{(2)}q_2x_{n+1}^2}{2!}+hy_n^{(1)}q_3x_{n+1}^3\right] \tag{17}$$

Substituting (13) – (17) into (12), we obtain our general integrator in (19),

$$y_{n+1} = \frac{\begin{gathered}y_n+hy_n^{(1)}+q_1x_{n+1}y_n+\frac{h^2y_n^{(2)}}{2!}+\ hy_n^{(1)}q_1x_{n+1}+q_2x_{n+1}^2y_n+\frac{h^3y_n^{(3)}}{3!}+\frac{h^2y_n^{(2)}q_1x_{n+1}}{2!}+hy_n^{(1)}q_2x_{n+1}^2\\ +q_3x_{n+1}^3y_n+\frac{h^4y_n^{(4)}}{4!}+\frac{h^3y_n^{(3)}q_1x_{n+1}}{3!}+\frac{h^2y_n^{(2)}q_2x_{n+1}^2}{2!}+hy_n^{(1)}q_3x_{n+1}^3+q_4x_{n+1}^4y_n\end{gathered}}{1+q_1x_{n+1}+q_2x_{n+1}^2+q_3x_{n+1}^3+q_4x_{n+1}^4}$$

Now, we let

$$\mathrm{A}=q_1x_{\mathrm{n+1}},\quad \mathrm{B}=q_2x_{n+1}^2,\quad \mathrm{C}=q_3x_{n+1}^3\ \text{ and }\ \mathrm{D}=q_4x_{n+1}^4. \tag{18}$$

Therefore, our general integrator formula for m = 4 becomes

$$y_{n+1}=\frac{\sum_{i=0}^{4}\frac{h^iy_n^{(i)}}{i!}+A\sum_{i=0}^{3}\frac{h^iy_n^{(i)}}{i!}+B\sum_{i=0}^{2}\frac{h^iy_n^{(i)}}{i!}+C\sum_{i=0}^{1}\frac{h^iy_n^{(i)}}{i!}+Dy_n}{1+A+B+C+D} \tag{19}$$

3 The convergence and consistency

For any numerical methods to be useful, such method must satisfy some basic properties. One- step methods like the stability of an inhomogeneous rational integrator for the solution or ordinary differential equations are normally described symbolically by

$$y_{n+1}=y_n+h\Phi(x_n,y_n;\ h), \tag{20}$$

where $\Phi(x_n,y_n;\ h)$ is called the increment function, x_n the mesh point and h the mesh size. [9] and also confirmed by [6] state that a rational integrator is said to be consistent if the increment function is consistent with the initial value problem (IVP) that is if $\Phi(\mathrm{x,y,0})=\mathrm{f(x,y)}$

Theorem: The stability of an inhomogeneous rational integrator for the solution of ordinary differential equations

$$y_{n+1}=\frac{\sum_{i=0}^{4}\frac{h^i y_n^{(i)}}{i!}+A\sum_{i=0}^{3}\frac{h^i y_n^{(i)}}{i!}+B\sum_{i=0}^{2}\frac{h^i y_n^{(i)}}{i!}+C\sum_{i=0}^{1}\frac{h^i y_n^{(i)}}{i!}+Dy_n}{1+A+B+C+D} \tag{21}$$

where the functions A, B, C and D are defined by equation and stated in equation (18) is convergent and consistent.

Proof:

Subtracting y_n from both sides of equ. (21), we have

$$y_{n+1}-y_n=\frac{\sum_{i=0}^{4}\frac{h^i y_n^{(i)}}{i!}+A\sum_{i=0}^{3}\frac{h^i y_n^{(i)}}{i!}+B\sum_{i=0}^{2}\frac{h^i y_n^{(i)}}{i!}+C\sum_{i=0}^{1}\frac{h^i y_n^{(i)}}{i!}+Dy_n}{1+A+B+C+D}-y_n$$

$$y_{n+1}-y_n=\frac{\sum_{i=1}^{4}\frac{h^i y_n^{(i)}}{i!}+A\sum_{i=1}^{3}\frac{h^i y_n^{(i)}}{i!}+B\sum_{i=1}^{2}\frac{h^i y_n^{(i)}}{i!}+Chy_n^{(i)}}{1+A+B+C+D}$$

$$\frac{y_{n+1}-y_n}{h}=\frac{y_n^{(i)}+\sum_{i=2}^{4}\frac{h^{i-1} y_n^{(i)}}{i!}+A\sum_{i=1}^{3}\frac{h^{i-1} y_n^{(i)}}{i!}+B\sum_{i=1}^{2}\frac{h^{i-1} y_n^{(i)}}{i!}+Chy_n^{(i)}}{1+A+B+C+D}$$

$$\operatorname*{Limit}_{h\to 0}\left(\frac{y_{n+1}-y_n}{h}\right):\ \operatorname*{Limit}_{x\to 0}\ A=\operatorname*{Limit}_{x\to 0}\ B=\operatorname*{Limit}_{x\to 0}\ C=\operatorname*{limit}_{x\to 0}\ D=0$$

$$\operatorname*{Limit}_{h\to 0}\left(\frac{y_{n+1}-y_n}{h}\right)=y_n^{(1)}$$

$\operatorname*{Limit}_{h\to 0}\left(\frac{y_{n+1}-y_n}{h}\right)=y_n^{(1)}=f(x_n,y_n)$ as required.

4 Stability considerations

To effectively solve initial value problems in ordinary differential equations which is stiff, we need numerical integrators that possess special stability properties such as inhomogeneous function. To test for stability, we apply the integrator to the test equation.

$$y'=\lambda y \tag{22}$$

where by induction, will give $y_n^{(k)}=\lambda^k y_n$ and $\lambda h=\bar{h}$. We have

$$A=\frac{\bar{h}}{-2},\quad B=\frac{3\bar{h}^2}{-28},\quad C=\frac{-\bar{h}^3}{84},\ \text{ and }\quad D=\frac{\bar{h}^4}{1680}. \tag{23}$$

Putting (23) into (21) we have

$$y_{n+1}=\frac{\begin{array}{c}y_n+hy_n^{(1)}+\frac{h^2y_n^{(2)}}{2}+\frac{h^3y_n^{(3)}}{6}+\frac{h^4y_n^{(4)}}{24}+Ay_n+Ahy_n^{(1)}+A\frac{h^2y_n^{(2)}}{2}+A\frac{h^3y_n^{(3)}}{6}+\\+By_n+Bhy_n^{(1)}+B\frac{h^2y_n^{(2)}}{2}+Cy_n+Chy_n^{(1)}+Dy_n\end{array}}{1+A+B+C+D}$$

$$y_{n+1}=\frac{\begin{array}{c}y_n+hy_n^{(1)}+\frac{h^2y_n^{(2)}}{2}+\frac{h^3y_n^{(3)}}{6}+\frac{h^4y_n^{(4)}}{24}-\frac{\bar{h}}{2}y_n-\left(\frac{\bar{h}}{2}\right)(\bar{h}y_n)-\left(\frac{\bar{h}}{2}\right)\frac{h^2y_n}{2}-(\frac{\bar{h}}{2})\frac{h^3y_n}{6}\\+\left(\frac{3\bar{h}^2}{28}\right)y_n+\left(\frac{3\bar{h}^2}{28}\right)hy_n+\left(\frac{3\bar{h}^2}{28}\right)\frac{h^2y_n}{2}-\left(\frac{\bar{h}^3}{84}\right)y_n-\left(\frac{\bar{h}^3}{84}\right)hy_n+\left(\frac{\bar{h}^4}{1680}\right)y_n\end{array}}{1+A+B+C+D}$$

$$y_{n+1}=\frac{\begin{array}{c}y_n+hy_n^{(1)}+\frac{h^2y_n^{(2)}}{2}+\frac{h^3y_n^{(3)}}{6}+\frac{h^4y_n^{(4)}}{24}-\frac{\bar{h}}{2}y_n-\left(\frac{\bar{h}}{2}\right)(\bar{h}y_n)-\left(\frac{\bar{h}^3}{4}\right)y_n-\left(\frac{\bar{h}^4}{12}\right)y_n\\+\left(\frac{3\bar{h}^2}{28}\right)y_n+\left(\frac{3\bar{h}^2}{28}\right)hy_n+\left(\frac{3\bar{h}^4}{56}\right)y_n-\left(\frac{\bar{h}^3}{84}\right)y_n-\left(\frac{\bar{h}^3}{84}\right)hy_n+\left(\frac{\bar{h}^4}{1680}\right)y_n\end{array}}{1+A+B+C+D}$$

$$y_{n+1}=\frac{\begin{array}{c}y_n+(\bar{h}-\frac{\bar{h}}{2})y_n+\left(\frac{\bar{h}^2}{2}-\frac{\bar{h}^2}{2}+\frac{3\bar{h}^2}{28}\right)y_n+\left(\frac{\bar{h}^3}{6}-\frac{\bar{h}^3}{4}+\frac{3\bar{h}^3}{28}-\frac{\bar{h}^3}{84}\right)y_n\\+(\frac{\bar{h}^4}{24}-\frac{\bar{h}^4}{12}+\frac{3\bar{h}^4}{56}-\frac{\bar{h}^4}{84}+\frac{\bar{h}^4}{1680})\end{array}}{1+A+B+C+D}$$

$$y_{n+1}=\frac{y_n+\frac{\bar{h}y_n}{2}+\frac{3\bar{h}^2y_n}{28}+\frac{\bar{h}^3y_n}{84}+\frac{\bar{h}^4y_n}{1680}}{1+A+B+C+D}$$

$$\frac{y_{n+1}}{y_n}=\frac{y_n+\frac{\bar{h}y_n}{2}+\frac{3\bar{h}^2y_n}{28}+\frac{\bar{h}^3y_n}{84}+\frac{\bar{h}^4y_n}{1680}}{1-\frac{\bar{h}}{2}+\frac{3\bar{h}^2}{28}-\frac{\bar{h}^3}{84}+\frac{\bar{h}^4}{1680}}.$$

Simplifying the numerator and the denominator, we get

$$\frac{y_{n+1}}{y_n}=\frac{\frac{1680+840\bar{h}+180\bar{h}^2+20\bar{h}^3+\bar{h}^4}{1680}}{\frac{1680-840\bar{h}+180\bar{h}^2-20\bar{h}^3+\bar{h}^4}{1680}}$$

$$\zeta(\bar{h})=\frac{y_{n+1}}{y_n}=\frac{1680+840\bar{h}+180\bar{h}^2+20\bar{h}^3+\bar{h}^4}{1680-840\bar{h}+180\bar{h}^2-20\bar{h}^3+\bar{h}^4} \qquad (24)$$

5 Analysis of the stability formular

Our new integrator is said to be Absolutely Stable if $\left|\zeta(\bar{h})\right| \leq 1$,

If we let $\zeta(\bar{h}) = \frac{1680+840\bar{h}+180\bar{h}^2+20\bar{h}^3+\bar{h}^4}{1680-840\bar{h}+180\bar{h}^2-20\bar{h}^3+\bar{h}^4} = \frac{\phi(\bar{h})}{\Psi(\bar{h})}$,
it implies $\left|\zeta(\bar{h})\right| \leq 1 \Leftrightarrow \left\|\frac{\phi(\bar{h})}{\Psi(\bar{h})}\right\| \leq 1$.

To anaylse the formula, we set $\bar{h} = u + iv$ where $i^2 = -1$. Hence $\|\phi(\bar{h})\| \leq \|\Psi(\bar{h})\| \therefore$ $\|\phi(\bar{h})\| - \|\Psi(\bar{h})\| \leq 0$,
where $\phi(\bar{h}) = 1680 + 840\bar{h} + 180\bar{h}^2 + 20\bar{h}^3 + \bar{h}^4$ and
$\Psi(\bar{h}) = 1680 - 840\bar{h} + 180\bar{h}^2 - 20\bar{h}^3 + \bar{h}^4$.

Let $\phi(\mathrm{u,v}) = A(u,v) + iB(u,v)$ and set $\bar{h} = u + iv$ where $i^2 = -1 \Psi(\mathrm{u,v}) = C(u,v) + iD(u,v)$
$\Rightarrow \phi(\mathrm{u,v}) = 1680 + 840(u+iv) + 180(u+iv)^2 + 20(u+iv)^3 + (u+iv)^4$
$\phi(\mathrm{u,v}) = 1680 + 840(u+iv) + 180(u^2 + 2uvi + v^2i^2) + 20(u^3 + 3u^2vi + 3uv^2i^2 + v^3i^3)$
$+(u^4 + 4u^3vi + 6u^2v^2i^2 + 4uv^3i^3 + v^4i^4)$.

On expansion, we get

$$A(\mathrm{u,v}) = u^4 - 6u^2v^2 + v^4 + 20u^3 - 60uv^2 + 180u^2 - 180v^2 + 840u + 1680$$
$$B(\mathrm{u,v}) = 4u^3v - 4uv^3 + 60u^2v - 20v^3 + 360uv + 840v$$

Hence, the inequality becomes
Simplifying further, the inequality above holds if and only if

$$A^2 = u^8 - 12u^6v^2 + 38u^4v^4 - 12u^2v^6 + v^8 + 40u^7 - 360u^5v^2 + 760u^3v^4 - 120uv^6$$
$$+760u^6 - 4920u^4v^2 + 6120u^2v^4 - 360v^6 + 8880u^5 - 38880u^3v^2 + 23280uv^4 +$$
$$69360u^4 - 185760u^2v^2 + 35760v^4 + 539600u^3 - 504000uv^2 + 1310400u^2 -$$
$$604800v^2 + 2822400u + 2822400$$

$$B^2 = 16u^6v^2 - 32u^4v^4 + 16u^2v^6 + 480u^5v^2 - 640u^3v^4 + 160uv^6 + 6480u^4v^2 - 5280u^2v^4$$
$$+400v^6 + 49920u^3v^2 - 21120uv^4 + 230400u^2v^2 - 33600v^4 + 604800uv^2 + 705600v^2$$
$$A^2 + B^2 = u^8 + 4u^6v^2 + 4u^2v^6 + v^8 + 40u^7 + 120u^5v^2 + 40uv^6 + 760u^6 + 1560u^4v^2$$
$$+840u^2v^4 + 40v^6 + 8880u^5 + 11040u^3v^2 + 2160uv^4 + 69360u^4 + 4460u^2v^2 +$$
$$2160v^4 + 369600u^3 + 100800uv^2 + 1310400u^4 + 100800v^2 + 2822400u + 2822400$$

$\Psi(\bar{h}) = 1680 - 840\bar{h} + 180\bar{h}^2 - 20\bar{h}^3 + \bar{h}^4.$

Set $\bar{h} = u + iv$ where $i^2 = -1$

Let $\Psi(\mathrm{u},\mathrm{v}) = C(u,v) + iD(u,v)$

$\Rightarrow \Psi(\mathrm{u},\mathrm{v}) = 1680 - 840(u+iv) + 180(u+iv)^2 - 20(u+iv)^3 + (u+iv)^4$

$\Psi(\mathrm{u},\mathrm{v}) = 1680 - 840(u+iv) + 180(u^2 + 2uvi + v^2i^2) - 20(u^3 + 3u^2vi + 3uv^2i^2 + v^3i^3)$
$+(u^4 + 4u^3vi + 6u^2v^2i^2 + 4uv^3i^3 + v^4i^4).$

On expansion, we get

$\|\Psi(u,v)\| = C^2(u,v) + D^2(u,v)$

$C(u,v) = u^4 - 6u^2v^2 + v^4 - 20u^3 + 60uv^2 + 180u^2 - 180v^2 - 840u + 1680$

$C^2 = u^8 - 12u^6v^2 + 38u^4v^4 - 12u^2v^6 + v^8 - 40u^7 + 360u^5v^2 - 760u^3v^4 + 120uv^6$
$+760u^6 - 4920u^4v^2 + 6120u^2v^4 - 360v^6 - 8880u^5 + 38880u^3v^2 - 23280uv^4 +$
$69360u^4 - 185760u^2v^2 + 35760v^4 - 369600u^3 + 504000uv^2 + 1310400u^2 -$
$604800v^2 - 2822400u + 2822400$

$D(u,v) = 4u^3v - 4uv^3 - 60u^2v + 20v^3 + 360uv - 840v$

$D^2 = 16u^6v^2 - 32u^4v^4 + 16u^2v^6 - 480u^5v^2 - 640u^3v^4 - 160uv^6 + 6480u^4v^2 - 5280u^2v^4$
$+400v^6 - 49920u^3v^2 + 21120uv^4 + 230400u^2v^2 - 33600v^4 - 604800uv^2 + 705600v^2$

$C^2 + D^2 = u^8 + 4u^6v^2 + 6u^4v^4 + 4u^2v^6 + v^8 - 40u^7 - 120u^5v^2 - 120u^3v^4 - 40uv^6 + 760u^6$
$+1560u^4v^2 + 840u^2v^4 + 40v^6 - 8880u^5 - 11040u^3v^2 - 2160uv^4 + 69360u^4 +$
$4460u^2v^2 + 2160v^4 - 369600u^3 - 100800uv^2 + 1310400u^4 + 100800v^2 - 2822400u$
$+2822400$

$\left|\left|\zeta(\bar{h})\right|\right| \leq 1 \iff \|\phi(u,v)\| - \|\Psi(u,v)\| \leq 0$

$\iff A(u,v)^2 + B(u,v)^2 - C(u,v)^2 - D(u,v)^2 \leq 0$

$A^2 + B^2 - (C^2 + D^2) = 80u^7 + 240u^5v^2 + 240u^3v^4 + 80uv^6 + 17760u^6$
$+22080u^3v^2 + 4320uv^4 + 739200u^3 + 201600uv^2 + 5644800u$
≤ 0

which shows that the integrator is $A -$ Stable if and only if $u < 0$.

To obtain our Region of Absolute Stability (RAS) we shall employ the polar form
where $u = R\cos\theta, and\ v = R\sin\theta$

$80R^7\cos(\theta)^7 + 240R^7\cos(\theta)^7\sin(\theta)^2 + 240R^7\cos(\theta)^7\sin(\theta)^4 + 80R^7\cos(\theta)\sin(\theta)^6$
$+17760R^5\cos(\theta)^5 + 22080R^5\cos(\theta)^3\sin(\theta)^2 + 4320R^5\cos(\theta)\sin(\theta)^4$
$+ 739200R^3\cos(\theta)^3$

$+201600R^3\cos(\theta)\,sin(\theta)^2 + 5644800R\cos(\theta)$
$\Rightarrow 80R^7cos(\theta)^7 + 240R^7cos(\theta)^7cos\theta(1-cos^2\theta) + 240R^7cos(\theta)^7cos\theta(1-cos^2\theta)^2$
$+80R^7\cos(\theta)cos\theta(1-cos^2\theta)^3 + 17760R^5cos(\theta)^5 + 22080R^5cos(\theta)^3cos\theta(1-cos^2\theta)$
$+4320R^5cos\theta(1-cos^2\theta)^2 + 739200R^3cos(\theta)^3 + 201600R^3\cos(\theta)\,cos\theta(1-cos^2\theta)^2$
$+5644800R\cos(\theta) = 0$
$\Rightarrow 80R^7cos(\theta)^7 - 20800R^5cos(\theta)^5 + 13440R^5cos(\theta)^3 + 22080R^5\cos(\theta)^7 + 4320R^5cos\theta$
$+3360000R^3cos(\theta)^3 + 201600R^3cos\theta + 2016000R^3cos(\theta)^5 + 5644800R\cos(\theta)$

6 The numerical analysis of the stability curve m = 4

The Jordan curve is the equation governing the region of absolute stability and the region of instability.

We take various degrees of θ from 0^0 to 360^0 and get a corresponding values of R.

Table 1.

θ^0	R	θ^0	R
0	25.0431937230128	195	-23.5878310205734
15	23.5878310205734	210	-20.2973921906952
30	20.2973921906952	225	-17.1429824862624
45	17.1429824862624	240	-15.9025813811573
60	15.9025813811578	255	-12.2683823503038
75	12.2683823503038	270	0.000000000000000
90	0.00000000000000	285	12.2683823503038
105	-12.2683823503038	300	15.90258138115783
120	-15.9025813811573	315	17.14298248626244
135	-17.1429824862624	330	20.29739219069527
150	-20.2973921906952	345	23.5878310205734
165	-23.5878310205734	360	25.0431937230128
180	-25.0431937230128		

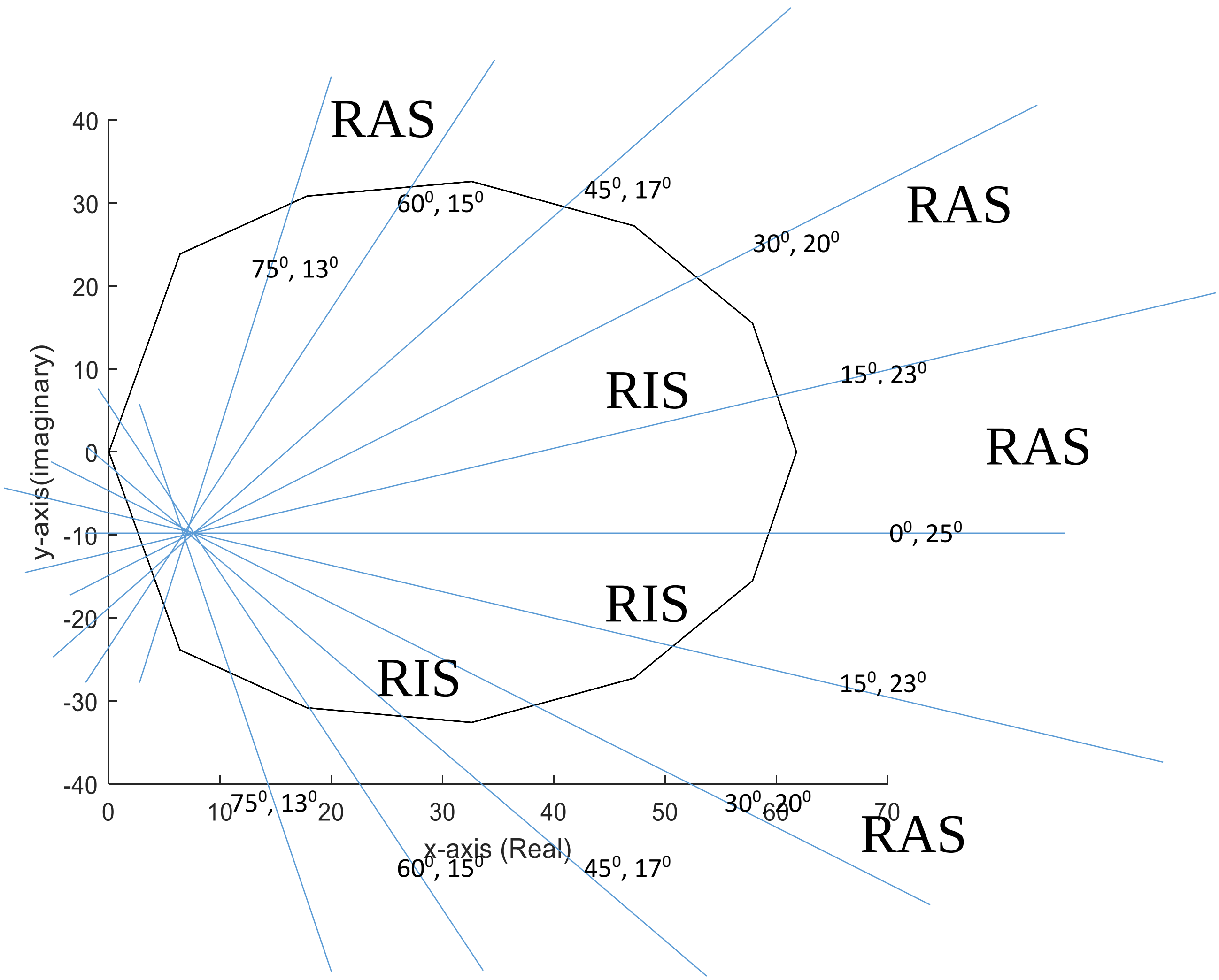

Figure 1.

Hence the integrator is A- stable so the Region of Absolute Stability of the integrator is the entire left – half of the complex plane and the space outside the diagram.

7 Numerical experiments

Problem 1: [10]: Consider the Riccati initial value problem
$y' = 1 + y^2, \; y(0) = 0$ with exact solution $y(x) = tanx,$ the step size $h = 0.1.$

Table 2

N	x_n	$y(x_n)$	Error in T_{n+1}	New method
1.0	0.000	1.0000000e+00	0.0000000e+00	0.0000000e+00
2.0	0.010	1.00359940e-01	2.5266803e-05	3.4086356e-05
3.0	0.020	2.02755448e-01	4.5446459e-05	4.0498433e-04
4.0	0.030	3.0939959e-01	6.3336523e-05	5.2880846e-04
5.0	0.040	4.2287746e-01	8.4243633e-05	7.4066631e-04
6.0	0.050	5.4642406e-01	1.2156595e-04	1.4395093e-04
7.0	0.060	6.8434715e-01	2.1034559e-04	2.4142834e-04
8.0	0.070	8.4273164e-01	4.4325944e-04	3.3508012e-04
9.0	0.080	1.0307117e-00	1.0731557e-04	2.2639649e-04
10.0	0.090	1.2629892e-00	2.8310182e-04	4.1641662e-04
11.0	0.100	1.5654375e-00	8.0297304e-03	7.0589025e-04

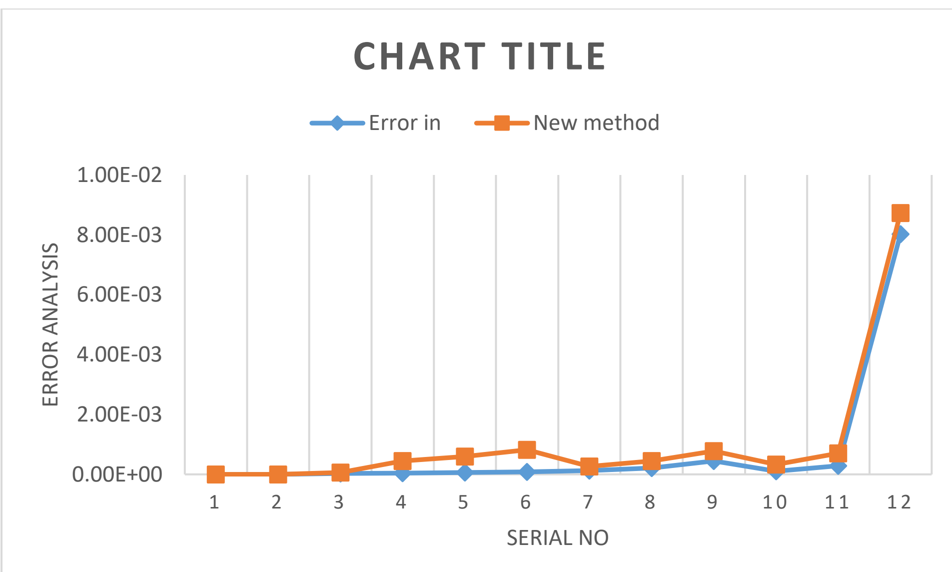

Figure 2

Remark: Table 2 and Figure 2 shows the performance of our new numerical integrator. Our new integrator competes favorably well with the development of a new one step scheme for the solution of initial value problem in ordinary differential equations of [10] with a very high rate of convergence.

Problem 2: [3]: Root Mean Square Runge-Kutta (RMS-RK4)
$y' = y^2 - y,\ y(0) = 2, 0 \le x \le 1$ and $h = 0.1$.

Table 3

N	TSOL	$y(x_n)$	Error in $RMS - RKM$	New method
0.1	2.2351E+00	2.2351E+00	1.4016E-05	3.40352e-05
0.2	2.5687E+00	2.5688E+00	5.6286E-05	3.08843e-05
0.3	3.0763E+00	3.0765E+00	1.9805E-04	3.76084e-05
0.4	3.9356E+00	3.9364E+00	7.8005E-04	2.49631e-05
0.5	5.6935E+00	5.6976E+00	4.0994E-03	2.49509e-04
0.6	1.1243E+01	1.1268E+01	2.4043E-02	2.91428e-04
0.7	-1.4543E+02	1.4498E+02	2.9041E+02	2.5608e-02
0.8	-8.8676E+00	4.8510E+15	4.8510E+15	2.2939e-01
0.9	-4.3516E+00	0.4406E+232	0.4406E+232	3.3641e+10
1.0	-2.7844E+00	NAN	NAN	NAN

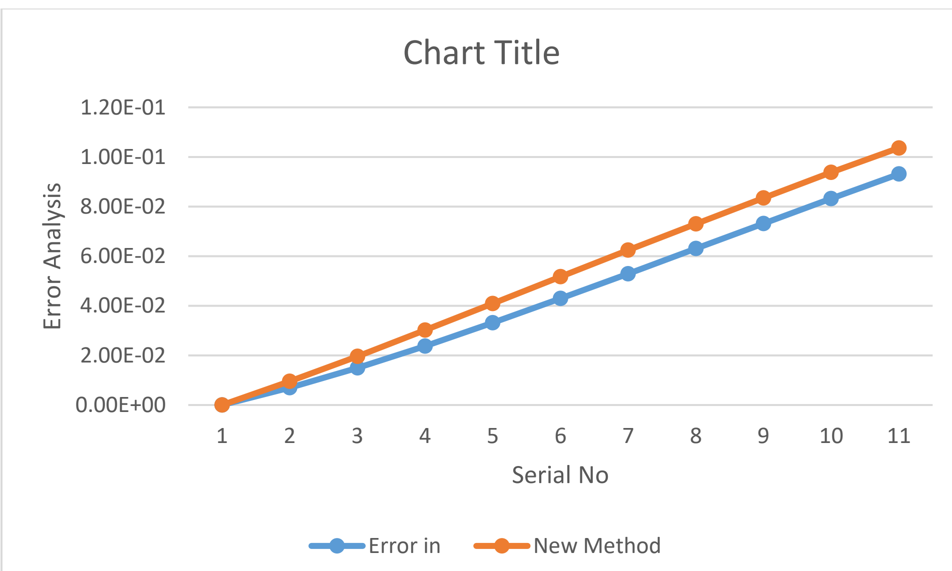

Figure 3.

Table 3 and Figure 3 shows the performance of our new numerical integrators. Our new integrator competes favorably well Implicit 4th Order RKF of [3], with a very high rate of convergence.

Problem 3: [12]

$y' = -sinx - 200(y - \cos x)\,; \qquad y(0) = 0; \quad h = 0.001.$

The exact solution is $y(x) = cosx - e^{-200x}$.

Table 4

XN	TSOL	Sunday, et al (2013)	New method
1.00e-03	1.81268e-01	1.8270e-07	4.1388e-09
2.00e-03	3.29677e-01	1.4085e-07	5.5915e-09
3.00e-03	4.51183e-01	5.5609e-07	5.5869e-09
4.00e-03	5.50663e-01	3.9270e-07	4.8676e-09
5.00e-03	6.32108e-01	2.2589e-07	3.8687e-09
6.00e-03	6.98787e-01	1.8561e-07	2.8328e-09
7.00e-03	7.53378e-01	1.5193e-07	1.8852e-09
8.00e-03	7.98071e-01	1.2601e-07	1.0808e-09
9.00e-03	8.34660e-01	1.1594e-07	4.3446e-10
10.0e+03	8.64614e-01	1.6619e-07	6.1043e-11

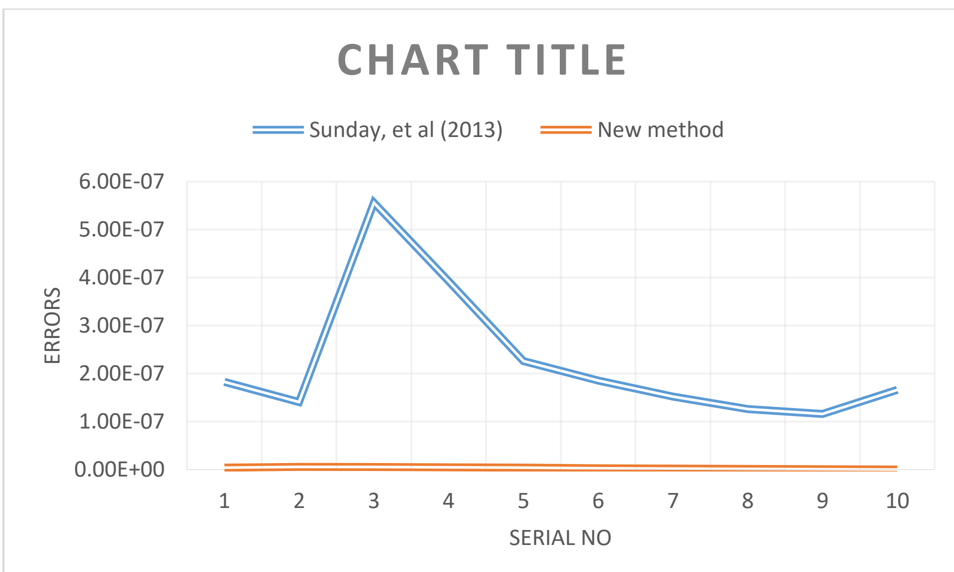

Figure 4.

Table 4 and Figure 4 shows the performance of our new numerical integrator. The trend shows that for a given mesh point, the errors in numerical integrator decreases as m increases. Our new integrator competes fairly well with the Extended Block Integrator method of [12].

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