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On The Applications of Type II Half Logistic Extended Exponential distribution to Modeling Failure data

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Abstract

This article introduces a three-parameter extension of the Exponentiated exponential distribution called Type II half logistic Extended Exponential distribution. The proposed model is quite flexible and adaptable to model any kind of life-time data. Its probability density function may sometimes be unimodal and its corresponding hazard rate may be of monotone or non-monotone shape. Standard statistical properties such as it ordinary and incomplete moments, quantile function, moment generating function, reliability function, order statistics, and Renyi entropy are obtained. The maximum likelihood method is used to obtain the estimates of the model parameters. Two practical examples of failure data sets are presented to demonstrate the flexibility of the developed model.

Keywords and phrases: maximum likelihood estimation; incomplete moments; hazard rate; type ii half logistic extended exponential distribution

1 Introduction

In a more recent time, several distributions have been developed and studied which serves life-time models in applied sciences. The most popular among the life-time model is the so-called generalized beta-g family of distributions developed and studied by [15], the others includes: Kumaraswamy generalized family was developed by [8], generalized transmuted-G family was introduced by [18], exponentiated generalized family studied by [9], the type I half-logistic family introduced by [11], Transformed-Transformer (T-X) developed by [2], Weibull-G family was introduced by [6], Exponentiated half-logistic family by [10], the Kumaraswamy Weibull family developed by [13], odd Lomax-G developed by [12], modified Kies-G family by [4], The Half-Logistic Generalized Weibull Distribution by [20] and generalized linear failure rate-G family was studied and developed by [3].

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The main motivation of the study is to further enhance the scope of applications of the Exponentiated Exponential distribution to cope with different shapes of the failure rate.

1.1 Extended exponential distribution

If the random variable X has the distribution function that follows the Extended Exponential (EE) distribution, then the corresponding density function is given as

$$g(x) = \lambda\theta e^{-\lambda x}\left(1 - e^{-\lambda x}\right)^{\theta-1}, \;\; x > 0; \lambda, \theta > 0, \tag{1}$$

and its cumulative density function and hazard function is respectively given by

$$G(x) = \left(1 - e^{-\lambda x}\right)^{\theta-1}, \;\; x > 0; \lambda, \theta > 0, \tag{2}$$

and

$$h(x) = \frac{\lambda\theta e^{-\lambda x}\left(1 - e^{-\lambda x}\right)^{\theta-1}}{1 - (1 - e^{-\lambda x})^{\theta-1}}, \tag{3}$$

where λ, a positive scale parameter and θ is a positive shape parameter. The shape of the hazard function of EE can take different forms, it has an increasing or decreasing hazard function $\theta > 1$ or $\theta < 1$ and for $\theta = 1$, it exhibits a constant hazard.

2 Type II Half logistic Extended Exponential distribution

The Type II half logistic-G (TIIHL-G) family of distributions was developed by [13] with the CDF given by

$$\begin{aligned} F(x; \theta, \varphi) &= 1 - \int_0^{-logG(x;\psi)} \frac{2\alpha e^{-\alpha t}}{(1 + e^{-\alpha t})^2} dt \\ &= \frac{2[F(x; \psi)]^\alpha}{1 + [F(x; \psi)]^\alpha}, \qquad x > 0; \alpha > 0, \end{aligned} \tag{4}$$

where $G(x; \varphi)$ the CDF of baseline model with parameter is vector φ and $G(x; \alpha, \varphi)$ is CDF obtained by using the T-X generator developed by [1]. The PDF of the TIIHL-G family is given by

$$f(x; \theta, \varphi) = \frac{2\theta g(x; \varphi)[G(x; \psi)]^{\alpha-1}}{[1 + [G(x; \psi)]^\alpha]^2}, \qquad x > 0, \theta > 0. \tag{5}$$

A random variable (r.v) X follows Type II half logistic Frechet ($TIIHLEE$) distribution if its PDF is obtained by inserting (1) and (2) in (5) as follows:

$$f(x) = \frac{2\alpha\lambda\theta e^{-\lambda x}\left(1 - e^{-\lambda x}\right)^{\theta\alpha-1}}{[1 + (1 - e^{-\lambda x})^{\theta\alpha}]^2}, \tag{6}$$

and its corresponding CDF is given by

$$F(x) = \frac{2\left(1 - e^{-\lambda x}\right)^{\theta\alpha}}{1 + (1 - e^{-\lambda x})^{\theta\alpha}}. \tag{7}$$

The graph of the CDF, PDF of TIIHLEE distribution is given in figure (1) and (2), respectively for various values of the parameters.

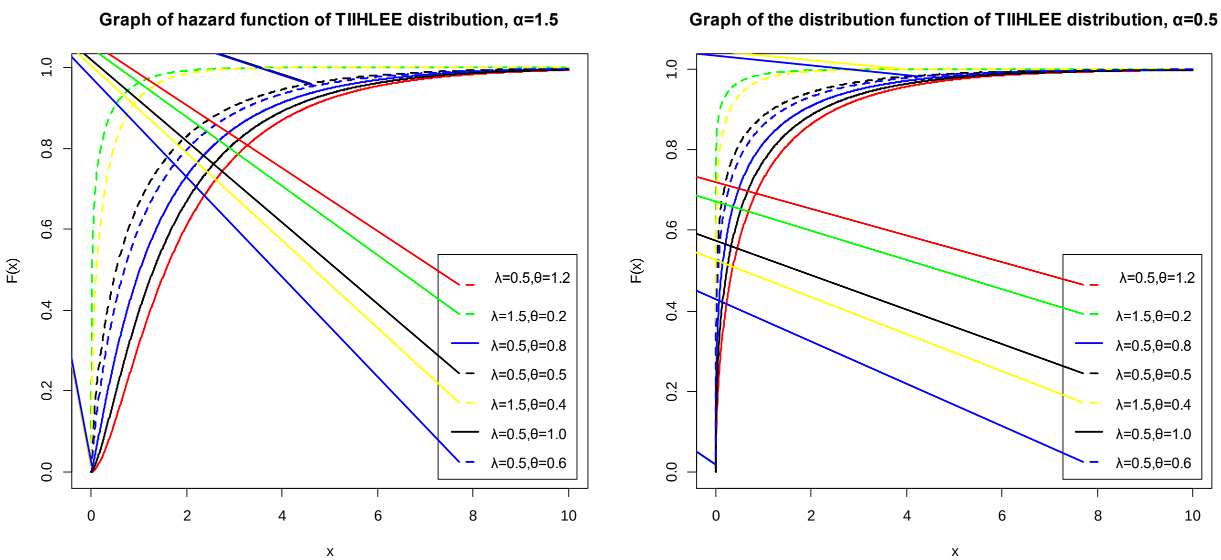

Figure 1: Graph of the PDF of TIIHLEE distribution

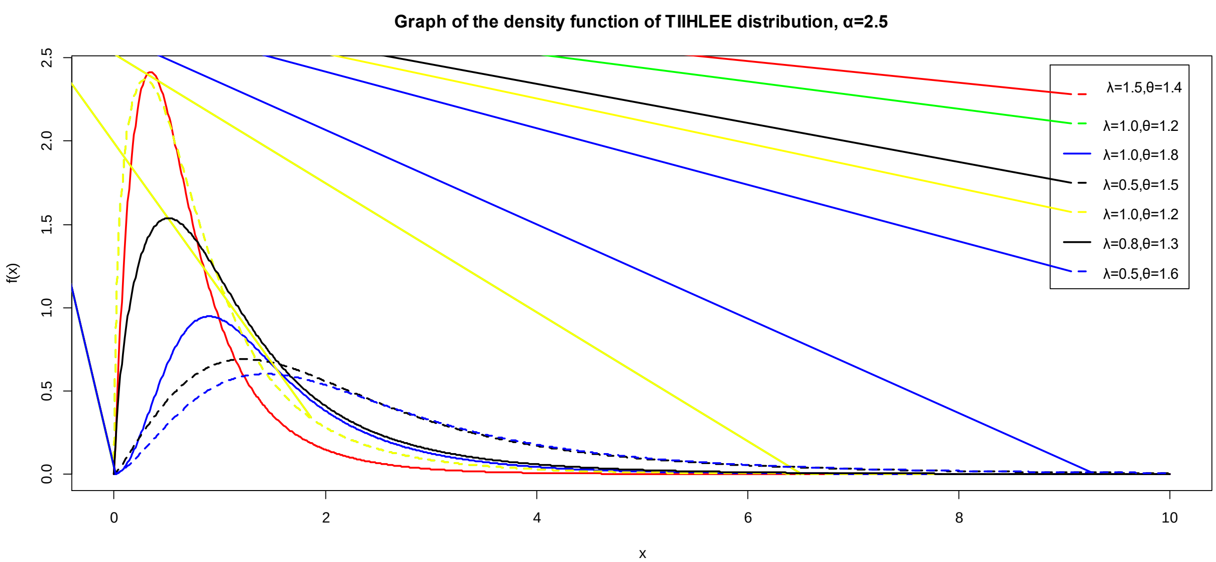

Figure 2: Graph of the PDF of TIIHLEE distribution

Correspondingly, an expression for the survival $(S(x))$ and the hazard $(h(x))$ function of TIIHLEE distribution is given respectively as

$$S(x) = \frac{1 - \left(1 - e^{-\lambda x}\right)^{\theta\alpha}}{1 + \left(1 - e^{-\lambda x}\right)^{\theta\alpha}}, \tag{8}$$

and

$$h(x) = \frac{2\alpha\lambda\theta e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\theta-1}\left(1-e^{-\lambda x}\right)^{\theta(\alpha-1)}}{[1-(1-e^{-\lambda x})^{\theta\alpha}][1+(1-e^{-\lambda x})^{\theta\alpha}]} \tag{9}$$

The graph of the $S(x)$, $h(x)$ of TIIHLEE distribution is given in figure (3) and (4) for various values of the parameters, respectively.

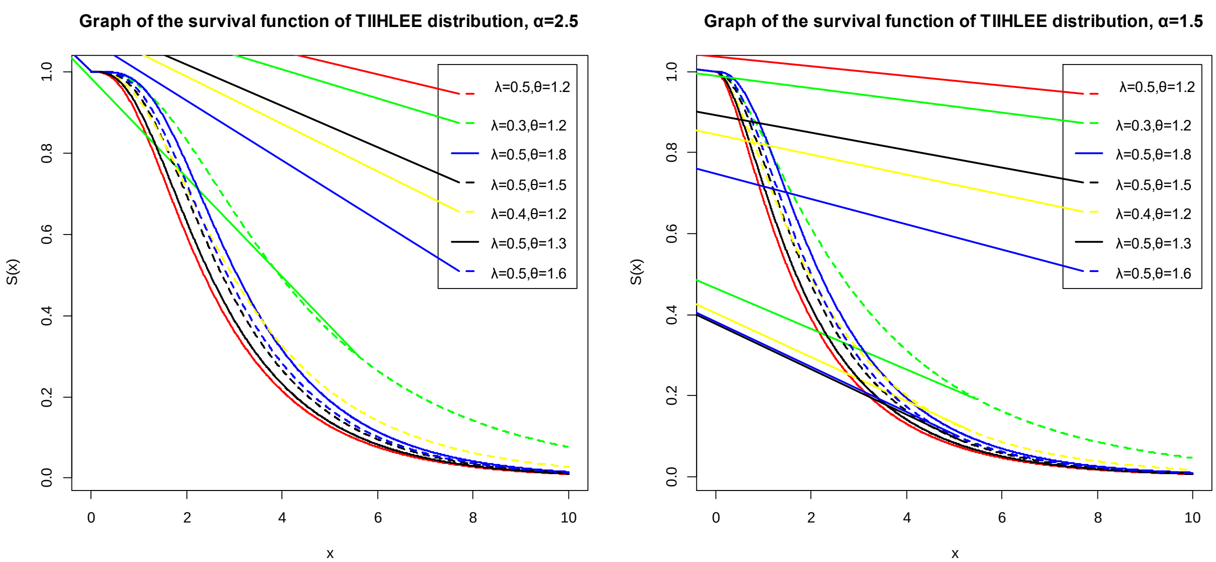

Figure 3: Graph of the $S(x)$ TIIHLEE distribution

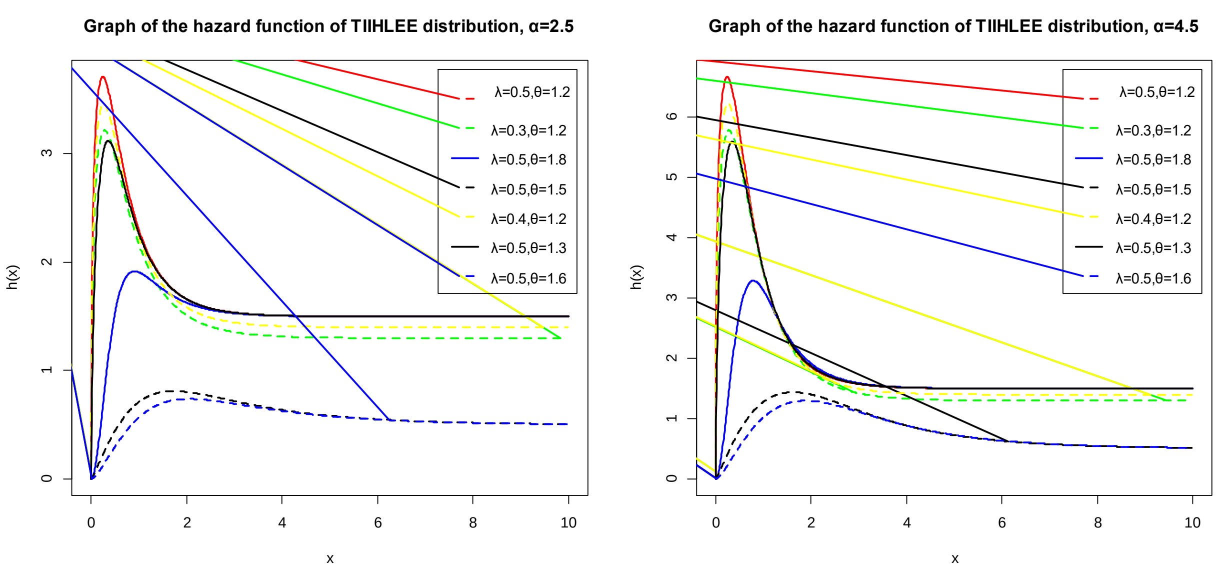

Figure 4: Graph of the $h(x)$ of TIIHLEE distribution

3 Important representations

In this section, an important tool for the expansion of the PDF for $TIIHLEE$ is provided. From the generalized binomial series given by

$$(1+v)^{-w} = \sum_{i=0}^{\infty} (-1)^i \binom{w+i-1}{i} v^i. \tag{10}$$

For $|v| < 1$ and w is a positive real non-integer. Then, by applying the binomial theorem (10) in (6), the density function of TIIHLEE distribution becomes

$$f(x) = 2\alpha\lambda\theta \sum_{i=0}^{\infty} (-1)^i \binom{i+1}{i} \binom{\theta\alpha(i+1)-1}{j} e^{-\lambda x(j+1)} \tag{11}$$

Here, we derived some properties of $TIIHLEE$ distribution.

3.1 Quantile function

The quantile function of X is denoted by $Q(u)$, defined as

$$Q(u) = -{}^{1}/_{\lambda}\left[1 - \frac{1}{\alpha\theta}\left(\frac{u}{2-u}\right)\right]. \tag{12}$$

By taking the value of $u = 0.5, 0.75,$ in (12) we obtain an expression for the median (m) and the upper quartile (UQ) of the TIIHLEE distribution respectively, given as

$$m = -{}^{1}/_{\lambda}\left[1 - \frac{1}{5\alpha\theta}\right], \tag{13}$$

$$QU = -{}^{1}/_{\lambda}\left[1 - \frac{3}{5\alpha\theta}\right]. \tag{14}$$

3.2 Moments of $TIIHLEE$ distribution

The moment plays an important role in distribution study and real data application(s). Now, we obtain the s^{th} moment of the $TIIHLEE$ distribution under certain regularity condition; the s^{th} moment of $TIIHLEE$ distribution is obtained as

$$\mu_s' = \int_0^{\infty} x^s f(x)dx = 2\alpha\lambda\theta \sum_{i=0}^{\infty} (-1)^i \binom{i+1}{i} \binom{\theta\alpha(i+1)-1}{j} J(x), \tag{15}$$

where

$$J(x) = \int_o^{\infty} x^s e^{-\lambda x(j+1)} dx. \tag{16}$$

Taking $= \lambda x(j+1)$, $x = \frac{w}{\lambda(j+1)}$, $dw = \lambda(j+1)dx$ and substitute in (16), we have

$$J(x) = [\lambda(j+1)]^{-(s+1)} \int_{o}^{\infty} w^{s} e^{-w} dw. \tag{17}$$

Finally, we have an expression for the kth moment of $TIIHLEE$ distribution given as

$$\mu_s' = 2\alpha\lambda\theta \sum_{i=0}^{\infty} (-1)^i \binom{i+1}{i} \binom{\theta\alpha(i+1)-1}{j} \left[\frac{1}{\lambda(j+1)}\right]^{s+1} \Gamma\,(s+1). \tag{18}$$

An expression for the mean (μ_s') the second (μ_2') and the third (μ_3') moments of $TIIHLEE$ distribution can be derived respectively as

$$\mu_1' = 2\alpha\lambda\theta \sum_{i=0}^{\infty} (-1)^i \binom{i+1}{i} \binom{\theta\alpha(i+1)-1}{j} \left[\frac{1}{\lambda(j+1)}\right]^{2}, \tag{19}$$

$$\mu_2' = 4\alpha\lambda\theta \sum_{i=0}^{\infty} (-1)^i \binom{i+1}{i} \binom{\theta\alpha(i+1)-1}{j} \left[\frac{1}{\lambda(j+1)}\right]^{3} \tag{20}$$

and

$$\mu_3' = 12\alpha\lambda\theta \sum_{i=0}^{\infty} (-1)^i \binom{i+1}{i} \binom{\theta\alpha(i+1)-1}{j} \left[\frac{1}{\lambda(j+1)}\right]^{4}, \tag{21}$$

the coefficient of skewness (q_s) and kurtosis (q_k) of $TIIHLF$ distribution are given by

$$k_s = \frac{\mu_3}{\left(\mu_2' - [\mu_1']^2\right)^{3/2}}, \qquad k_k = \frac{\mu_4}{\left(\mu_2' - [\mu_1']^2\right)^{2}}.$$

Table 1 gives the numerical values of mean (μ_1'), variance (σ^2), skewness (k_s) and kurtosis (k_k) for various values of the parameters of $TIIHLEE$ distribution.

Table 1: Table of first six moments, variance, skewness and kurtosis of TIIHLEE distribution.

	$\theta = 0.2, \lambda = 1.2$				
Moments	$\alpha = 0.1$	$\alpha = 0.4$	$\alpha = 0.8$	$\theta = 1.2$	$\theta = 1.6$
μ_1	0.0135	0.0563	0.1191	0.1887	0.2650
μ_2	0.0166	0.0674	0.1372	0.2095	0.2843
μ_3	0.0375	0.1508	0.3038	0.4590	0.6165
μ_4	0.1199	0.4808	0.9646	1.4515	1.9414
μ_5	0.4904	1.9638	3.9334	5.9087	7.8897
μ_6	2.4310	9.7289	19.4712	29.2269	38.9961
σ^2	0.0164	0.0642	0.1230	0.1739	9.2141
k_s	17.5090	8.5864	5.9833	4.8796	4.3180
k_k	437.382	108.6142	54.9103	37.8987	30.3941

3.3 Incomplete moments of TIIHLEE distribution

The incomplete moment of $TIIHLEE$ distribution can be obtained using (11) as

$$\eta_s' = \int_0^t x^s f(x)dx = 2\alpha\lambda\theta \sum_{i=0}^{\infty} (-1)^i \binom{i+1}{i} \binom{\theta\alpha(i+1)-1}{j} J(x) \tag{22}$$

where

$$J(x) = \int_o^t x^s e^{-\lambda x(j+1)} dx. \tag{23}$$

Taking $= \lambda x(j+1)$, $x = \frac{w}{\lambda(j+1)},\ dw = \lambda(j+1)dx$ and substitute in (23), we have

$$J(x) = [\lambda(j+1)]^{-(s+1)} \int_o^{\lambda x(j+1)} w^s e^{-w} dw\ . \tag{24}$$

Finally, we have an expression for the kth moment of $TIIHLEE$ distribution given as

$$\eta_s' = 2\alpha\lambda\theta \sum_{i=0}^{\infty} (-1)^i \binom{i+1}{i} \binom{\theta\alpha(i+1)-1}{j} \left[\frac{1}{\lambda(j+1)}\right]^{s+1} \Gamma\left(s+1, \lambda t(j+1)\right) \tag{25}$$

3.4 Renyi entropy

Renyi entropy was proposed by [17]. It can be obtained by

$$I_\eta(x) = \frac{1}{1-\eta} logV \tag{26}$$

where

$$V = \int_{-\infty}^{\infty} f^\eta(x)dx, \qquad \eta > 0, \eta \neq 1. \tag{27}$$

Plugging (6) in (27), then applying binomial theorem given in (10), we have

$$V = 2^\eta \alpha^\eta \lambda^\eta \theta^\eta \sum_{I,J=0}^{\infty} (-1)^{i+j} \binom{2\eta+i-1}{i} \binom{\theta\alpha(\eta+i)-\eta}{j} \int_0^{\infty} e^{-\lambda x(\eta+j)} dx \tag{28}$$

$$= 2^\eta \alpha^\eta \lambda^\eta \theta^\eta \sum_{I,J=0}^{\infty} (-1)^{i+j} \binom{2\eta+i-1}{i} \binom{\theta\alpha(\eta+i)-\eta}{j} [\theta(\eta+j)]^{-1}.$$

Finally, we have an expression for the Renyi entropy of $TIIHLEE$ distribution as

$$I_\eta(x) = \frac{1}{1-\eta} log\left(2^\eta \alpha^\eta \lambda^\eta \theta^\eta \sum_{I,J=0}^{\infty} (-1)^{i+j} \binom{2\eta+i-1}{i} \binom{\theta\alpha(\eta+i)-\eta}{j} [\theta(\eta+j)]^{-1}\right) \tag{29}$$

3.5 Order statistics of TIIHLEE distribution

Order statistics is widely applied in statistical theory. Suppose $X_1, X_2, \ldots, X_n$ be a random sample having CDF $F(x)$. Let $x_{1:1}, x_{2:n}, \ldots, x_{n:n}$ be the ordered sample of size n, then the density of r^{th} order statistics is given as

$$f_{r:n}(x) = w \sum_{l=0}^{n-r} (-1)^l \binom{n-r}{l} f(x) F(x)^{l+r-1} \tag{30}$$

where $w = \frac{n!}{(n-r)!r!}$. The PDF of the r^{th} order statistics of $TIIHLEE$ distribution is obtained by putting (6) and (7) in (30), changing s with $l + r - 1$ followed by simple algebraic manipulation, we have

$$f_{r:n}(x) = 4\alpha\lambda\theta w \sum_{l=0}^{n-r} (-1)^l \binom{n-r}{l} e^{-\lambda x} \left(1 - e^{-\lambda x}\right)^{\theta\alpha(l+r)} \left[1 + \left(1 - e^{-\lambda x}\right)^{\theta\alpha}\right]^{-(l+r+3)}. \tag{31}$$

By taking the value of $r = 1$ and n in equation (31), we respectively obtain the first and the n[th] order statistics of $TIIHLEE$ distribution.

3.6 Maximum likelihood estimation

Given a random sample of $x_1, x_2, \ldots, x_n$ from the TIIHLEE distribution, the likelihood function for $\psi = (\alpha, \lambda, \theta)$ is

$$l(\alpha, \lambda, \theta) = log(2\alpha\lambda\theta) + \lambda \sum_{i=1}^{n} x_i + (\theta\alpha - 1) \sum_{i=1}^{n} log\left(1 - e^{-\lambda x_i}\right) - 2\theta \sum_{i=1}^{n} log\left[1 + \left(1 - e^{-\lambda x_i}\right)^{\theta\alpha}\right]. \tag{32}$$

Maximizing $l(\alpha, \lambda, \theta)$ with respect to $\alpha, \lambda,$ and θ, we obtain the following system of nonlinear equations:

$$\frac{n}{\alpha} + \theta \sum_{i=1}^{n} log\left(1 - e^{-\lambda x_i}\right) - 2 \sum_{i=1}^{n} \left(\frac{\left(1 - e^{-\lambda x_i}\right)^{\theta\alpha} log\left(1 - e^{-\lambda x_i}\right)}{[1 + (1 - e^{-\lambda x_i})^{\theta\alpha}]}\right) = 0 \tag{33}$$

$$\frac{n}{\lambda} - \sum_{i=1}^{n} \frac{x_i e^{-\lambda x_i}}{(1 - e^{-\lambda x_i})} + 2 \sum_{i=1}^{n} \frac{x_i e^{-\lambda x_i}}{(1 - e^{-\lambda x_i})[1 + (1 - e^{-\lambda x_i})^{\theta\alpha}]} = 0 \tag{34}$$

$$\frac{n}{\theta} + \alpha \sum_{i=1}^{n} log\left(1 - e^{-\lambda x_i}\right) - 2\alpha \sum_{i=1}^{n} \frac{log\left(1 - e^{-\lambda x_i}\right)}{[1 + (1 - e^{-\lambda x_i})^{\theta\alpha}]} = 0 \tag{35}$$

respectively. The solution of this system has to be calculated numerically to get the estimates $\hat{\alpha}_{MLE}$, $\hat{\lambda}_{MLE}$ and $\hat{\theta}_{MLE}$.

4 Application

In this section, we compare the results of fitting the TIIHLEE (α, λ, θ) and EE (λ, θ) and E (λ) distributions to two real data sets. All the computations done using the R programming language (R Development Core Team, First, we consider the data set consisting of the length of intervals times at which vehicles pass a point on a road. The first data set is given as follows: 2.50, 2.60, 2.60, 2.70, 2.80, 2.80, 2.90, 3.00, 3.00, 3.10, 3.20, 3.40, 3.70, 3.90, 3.90, 3.90, 4.60, 4.70, 5.00, 5.60, 5.70, 6.00, 6.00, 6.10, 6.60, 6.90, 6.90, 7.30, 7.60, 7.90, 8.00, 8.30, 8.80, 8.80, 9.30, 9.40, 9.50, 10.1, 11.0, 11.3, 11.9, 11.9, 12.3, 12.9, 12.9, 13.0, 13.8, 14.5, 14.9, 15.3, 15.4, 15.9, 16.2, 17.6, 20.1, 20.3, 20.6, 21.4, 22.8, 23.7, 23.7, 24.7, 29.7, 30.6, 31.0, 34.1, 34.7, 36.8, 40.1, 40.2, 41.3, 42.0, 44.8, 49.8, 51.7, 55.7, 56.5, 58.1, 70.5, 72.6, 87.1, 88.6, 91.7, 119.8. Total Time on Test plot is given in Figure 3a which indicates that the data exhibits non-monotone failure rate and Figure (3b) is the graph of empirical density function for data I, shows that the data is positively skewed. From Table 2.0, it could be observed that the data is over-dispersed leptokurtic.

The second data set introduced by [14]: 112, 68, 84, 109, 153, 143, 60, 70, 98, 164, 63, 63, 77, 91, 91, 66, 70, 77, 63, 66, 66, 94, 101, 105, 108, 112, 115, 126, 161, 178. These data represent the numbers of tumor-days of 30 rats fed with unsaturated diet. [5] and [7] used the Gompertz distribution for these data set in order to obtain exact confidence intervals and joint confidence regions for the parameters based on two different statistical analysis. Figure 4 indicates that the second data set exhibits increasing failure rate and moderately positive skewed. From Table 2.0, it could be observed that the data is over-dispersed mesokurtic.

The MLEs of the model parameters errors in parentheses) and the values of the AIC (Akaike Information CAIC (Consistent Akaike Information Criterion) and BIC (Bayesian Information Criterion)

Table 2 : Exploratory data analysis for data set I

min	q_1	median	Mean	q_3	Max.	variance	kurtosis	Skewness	Range
2.5	5.9	12.1	21.6	29.9	119.8	574.6	6.6	1.9	117.3

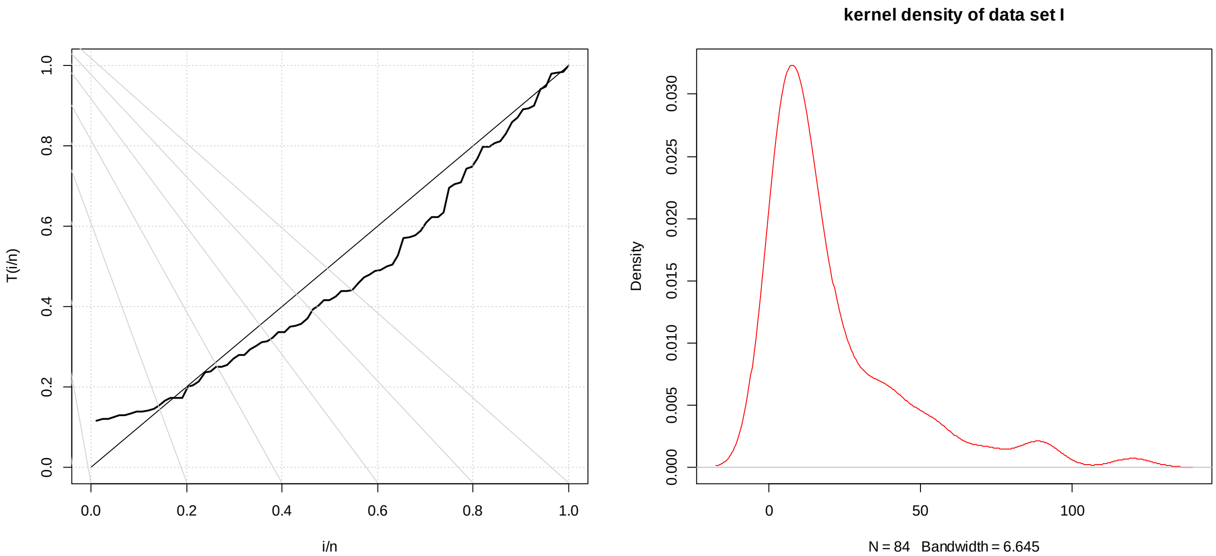

Figure 5. TTT plot and empirical density plots for data set 1

Table 3.0 MLEs, standard error (in braces) and Measures of goodness of fit for the parameters of the models for dataset I

	Parameters Estimates			*Measures of goodness of fit*			
Model	α	λ	θ	l	*AIC*	*BIC*	*CAIC*
TIIHLEE	12.76 (1.33)	0.01 (0.00)	0.52 (0.05)	53.96	113.93	121.22	114.23
EE	– (–)	1.11 (3.35)	0.04 (0.13)	342.12	688.23	693.09	688.38
E	– (–)	0.0 (0.01)	– (–)	342.12	686.23	688.66	686.28

Table 4: Exploratory Data Analysis for data set II

Min	q_1	median	Mean	q_3	Max.	variance	kurtosis	Skewness	Range
60	68.5	92.5	98.48	112.0	178.0	1150.5	2.7	0.8	118.0

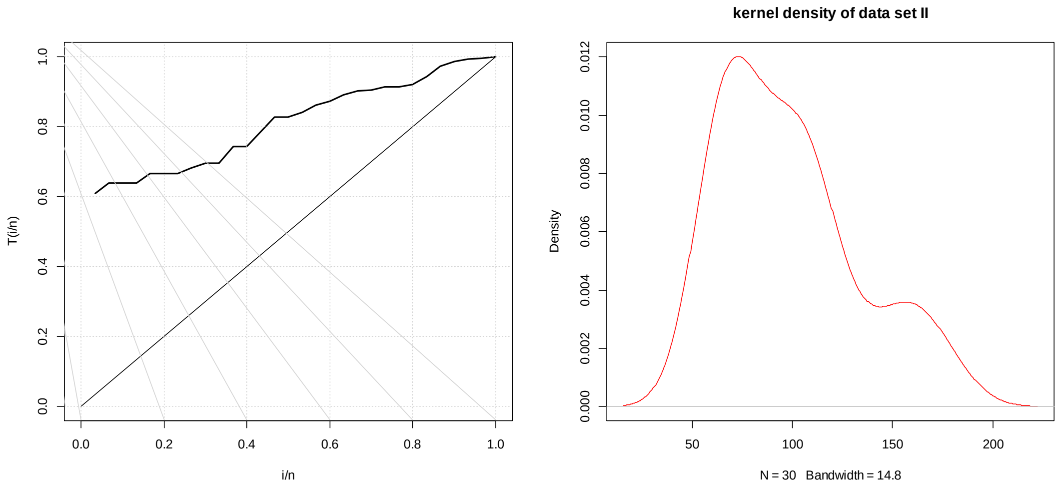

Figure 6. TTT plot and empirical density plots for data set II

Table 5.0 MLEs, standard error (in braces) and Measures of goodness of fit for the parameters of the models for dataset II

	Parameters Estimates			*Measures of goodness of fit*			
Model	α	λ	θ	l	*AIC*	*BIC*	*CAIC*
TIIHLEE	8.34 (1.37)	0.01 (0.00)	0.94 (0.20)	84.89	175.78	179.98	176.70
EE	− (−)	0.02 (0.00)	5.44 (1.51)	149.65	303.30	306.10	303.74
E	− (−)	0.03 (0.01)	− (−)	155.45	312.90	314.40	313.03

5 Conclusion

In this paper, we studied and developed a new generalization of extended exponential distributions called Type II half-logistic Extended Exponential distribution, which is developed based using Type II half-logistic-G family. The Type II half-logistic Exponentiated Exponential distribution generalizes the Exponentiated Exponential and Exponential distribution. Some properties of the newly developed distribution such as moments, incomplete moments, reliability and hazard rate functions, order statistics, Renyi entropy, and moment generating functions are derived. The maximum likelihood estimators are used to obtain the estimates of the parameters of the distribution. An application of the *TIIHLEE* distribution to two lifetime data sets shows that the new distribution provides a reasonable fit than other types of models when used to model life-time data.

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