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A family of non-linear rational integrators for the solution in ordinary differential equations

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Abstract

In this research work, we derived a Family of Non-Linear rational integrators for the solution of ordinary differential equations. We establish the various integrator formula with an interpolants of order m = 2 and 3 respectively with the use of MAPLE-18 and MATLAB softwares. Our method was compared with an explicit fourth-stage fourth-order Runge-Kutta methods and the Extended Block Integrator for First-Order Stiff and Oscillatory Differential Equations. Our results show good improvement over these recent existing methods compared with, and also ascertain the convergence and consistency of our scheme. Our result shows that our integrator is stable analytically and computationally.

Keywords and Phrases: rational integrator; stiff; singular and oscillatory problems; stability function and region of absolute stability (RAS)

1 Introduction

This research work is concerned with the determination of solution of initial value problems in Ordinary Differential Equations (ODEs). Many mathematical models have been formulated in terms of the rate of change of one or more physical variables and these naturally lead to differential equations, [5], [7]. The basis for many numerical integrators used in solving the differential equation in finding the solution of initial value problems in the form:

$$y' = f(x, y), \quad y(x_0) = y_0, \ \ a \le x \le b, \tag{1}$$

where $f(x, y)$ must satisfy a Lipschitz condition with respect to y and is defined and continuous in a region $D \subset [a,b]$ that are either Stiff or Singular.

Lipschitz constant according to [2], from (1) above, $f(x, y)$ is said to satisfy a Lipschitz condition in y, over the region D, if there exist a constant L such that

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$$||f(x, y_1) - f(x, y_2)|| \leq L||y_1 - y_2||, \tag{2}$$

where L is called the Lipschitz constant and $f(x, y)$ is said to be Lipschitzian. By virtue of the relation.

Definition 1: Stiffness [6] The system (1) is said to be stiff over the interval [a,b] if for every $x \epsilon [a, b]$, the eigenvalues {λ,(x), s = 1,…, m} of the Jacobian matrix

$$J = \frac{\partial f}{\partial y} \tag{3}$$

satisfy the following conditions:

Re $\lambda_s(x) < 0, s = 1, \dots, m$

where Re $\lambda_s(x)$ represents the real component of the complex number $\lambda_s(x)$

Stiffness Ratio = $\frac{\lambda_r(x)}{\lambda_r(x)}$>> 1, r,s = 1,…,m.

For the linear initial value problem

$$y' = Ay + g, \qquad y(x_0) = y_0$$

$$\frac{\partial f}{\partial y} = \frac{\partial f'}{\partial y} = A$$

is m x m matrix and $g \epsilon R^m$.

Definition 2: [7]: A numerical method is said to be A-stable if its region of absolute stability contains the whole of the left-hand half-plane $\text{Re } h\lambda < 0$.
However, A-stable is a severe requirement to ask of a numerical method for stiffness to be satisfied.

Definition 3: [11]: A numerical method is said to be stiffly-stable if (i) the region of absolute stability contains $\Re_1$ and $\Re_1$ and (ii) it is accurate for all $h \in \Re_2$ when applied to the test equation $y' = \lambda y, \lambda$ a complex constant with $\text{Re } h\lambda < 0$, where

$\Re_1 = \{[h\lambda]\text{Re } h\lambda < -a\}$,

$\Re_2 = \{[h\lambda] - a \leq \text{Re } h\lambda \leq b, -c \leq \text{Im}h\lambda \leq c\}$,

and a, b and c are positive constants.

2 Derivation of the method

The rational interpolant U : R → R is defined by

$$y_{n+1} = r + \frac{\sum_{i=0}^{m-1} p_i x_{n+1}^i}{1 + \sum_{i=1}^{m} q_i x_{n+1}^i} \tag{4}$$

where p_i *and* q_i are called the integrator parameters, and r is an inhomogeneous constant.

Case 1: At m = 2, we have

$$y_{n+1} = r + \frac{\sum_{i=0}^{1} p_i x_{n+1}^i}{1 + \sum_{i=1}^{2} q_i x_{n+1}^i} \tag{5}$$

where $p_{0,}\ p_{1,}$ *and* $q_{1,}\ q_{2,}$ are called the integrator parameters and r is an inhomogeneous constant. By the use of binomial expansion for rational functions and employing the Taylor's series expansion on y_{n+1}, as

$$y_{n+1} = \sum_{k=0}^{\infty} \frac{h^k y_n^{(k)}}{k!}. \tag{6}$$

Comparing coefficients in terms of h, and putting in matrix form, we have

$$\begin{bmatrix} \frac{y_n^{(3)}}{3!(n+1)^3} & \frac{y_n^{(2)}}{2!(n+1)^2} \\ \frac{y_n^{(2)}}{2!(n+1)^2} & \frac{y_n^{(1)}}{(n+1)} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = - \begin{bmatrix} \frac{y_n^{(4)}}{4!(n+1)^4} \\ \frac{y_n^{(3)}}{3!(n+1)^3} \end{bmatrix} \tag{7}$$

Applying the Crammers rule to determine q_1, q_2 we have

where $q_i = \frac{x_i}{|A|}$, where $i = 1,2$ (8)

We employ Maple 18 software compiler and obtained the determinant of A, x_1 and x_2 respectively

$$|A| = \begin{bmatrix} \frac{y_n^{(3)}}{3!(n+1)^3} & \frac{y_n^{(2)}}{2!(n+1)^2} \\ \frac{y_n^{(2)}}{2!(n+1)^2} & \frac{y_n^{(1)}}{(n+1)} \end{bmatrix} \tag{9}$$

$$|A| = (n+1)2y_n^{(3)}y_n^{(1)} - 3{y_n^{(2)}}^2$$

$$|x_1| = \begin{bmatrix} -\frac{y_n^{(4)}}{4!(n+1)^4} & \frac{y_n^{(2)}}{2!(n+1)^2} \\ -\frac{y_n^{(3)}}{3!(n+1)^3} & \frac{y_n^{(1)}}{(n+1)} \end{bmatrix} \tag{10}$$

$$|x_1| = 2y_n^{(2)}y_n^{(3)} - y_n^{(4)}y_n^{(1)}$$

$$|x_2| = \begin{bmatrix} \frac{y_n^{(3)}}{3!(n+1)^3} & -\frac{y_n^{(4)}}{4!(n+1)^4} \\ \frac{y_n^{(2)}}{2!(n+1)^2} & -\frac{y_n^{(3)}}{3!(n+1)^3} \end{bmatrix} \quad (11)$$

$$|x_2| = 4y_n^{(3)^2} - 3y_n^{(4)}y_n^{(2)}.$$

At the integration point$x = x_{n+1}$,

$$U(x_{n+1}) = \sum_{k=0}^{\infty} c_k\, x_{n+1}^k. \quad (12)$$

With these definitions for $c_r s$, the determinant |A| can be written as follows:

$$|A| = \frac{h^4}{x_{n+1}^4}\left[(n+1)[2y_n^{(3)}y_n^{(1)} - 3y_n^{(2)^2}\right], \quad (13)$$

$$|x_1| = \frac{h^5}{2x_{n+1}^5}\left[2y_n^{(2)}y_n^{(3)} - y_n^{(4)}y_n^{(1)}\right], \quad (14)$$

$$|x_2| = \frac{h^6}{-12x_{n+1}^6}\left[4y_n^{(3)^2} - 3y_n^{(4)}y_n^{(2)}\right]. \quad (15)$$

Since $q_i{}'s$ are defined by (8), we have

$$q_1 x_{n+1}^1 = \frac{h}{2}\frac{\left[2y_n^{(2)}y_n^{(3)} - y_n^{(4)}y_n^{(1)}\right]}{\left[(n+1)[2y_n^{(3)}y_n^{(1)} - 3y_n^{(2)^2}\right]}, \quad (16)$$

$$q_2 x_{n+1}^2 = \frac{h^2}{-12}\frac{\left[4y_n^{(3)^2} - 3y_n^{(4)}y_n^{(2)}\right]}{\left[(n+1)2y_n^{(3)}y_n^{(1)} - 3y_n^{(2)^2}\right]} \quad (17)$$

Now, recall from (4),

$$\begin{aligned} y_n &= r + p_0 \\ r &= y_n - p_0 \quad (18) \\ p_1 &= \frac{hy_n^{(1)}}{1!\,x_{n+1}} + p_0 q_1 \\ p_1 x_{n+1} &= hy_n^{(1)} + p_0 q_1 x_{n+1} \quad (19) \\ p_0 q_2 &= \frac{h^2 y_n^{(2)}}{2!x_{n+1}^2} + \frac{hy_n^{(1)}}{1!x_{n+1}} q_1 \end{aligned}$$

$$p_0 = \frac{-1}{q_2}\left[\frac{h^2 y_n^{(2)} q_1 x_{n+1}}{2!} + h y_n^{(1)} q_1 x_{n+1}\right] \tag{20}$$

$$r = y_n - \frac{1}{q_2}\left[\frac{h^2 y_n^{(2)} q_1 x_{n+1}}{2!} + h y_n^{(1)} q_1 x_{n+1}\right]. \tag{21}$$

Putting equation (16 – 21) into 5, our integrator formula for m = 2 becomes

$$y_{n+1} = \frac{\sum_{i=0}^{2} \frac{h^i y_n^{(i)}}{i!} + A \sum_{i=0}^{1} \frac{h^i y_n^{(i)}}{i!} + B y_n}{1 + A + B}. \tag{22}$$

Case 2: At m = 3, from (4) we have

$$y_{n+1} = r + \frac{\sum_{i=0}^{2} p_i x_{n+1}^i}{1 + \sum_{i=1}^{3} q_i x_{n+1}^i} \tag{23}$$

where p_0, p_1, p_2 and q_1, q_2, q_3 are called the integrator parameters and r is an inhomogeneous constant. By the use of binomial expansion for rational functions and employing the Taylor's series expansion on y_{n+1}, as $y_{n+1} = \sum_{k=0}^{\infty} \frac{h^k y_n^{(k)}}{k!}$ (24)

Comparing coefficients in terms of h, and putting in matrix form, we have

$$\begin{bmatrix} \frac{y_n^{(5)}}{5!(n+1)^5} & \frac{y_n^{(4)}}{4!(n+1)^4} & \frac{y_n^{(3)}}{3!(n+1)^3} \\ \frac{y_n^{(4)}}{4!(n+1)^4} & \frac{y_n^{(3)}}{3!(n+1)^3} & \frac{y_n^{(2)}}{2!(n+1)^2} \\ \frac{y_n^{(3)}}{3!(n+1)^3} & \frac{y_n^{(2)}}{2!(n+1)^2} & \frac{y_n^{(1)}}{(n+1)} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = -\begin{bmatrix} \frac{y_n^{(6)}}{6!(n+1)^6} \\ \frac{y_n^{(5)}}{5!(n+1)^5} \\ \frac{y_n^{(4)}}{4!(n+1)^4} \end{bmatrix} \tag{25}$$

Applying the Crammers rule to determine q_1, q_2, q_3 we have
where $q_i = \frac{x_i}{|A|}$, where $i = 1,2,3$ (26)
we employ Maple 18 software compiler and obtained

The determinant of A

$$|A| = \begin{bmatrix} \frac{y_n^{(5)}}{5!(n+1)^5} & \frac{y_n^{(4)}}{4!(n+1)^4} & \frac{y_n^{(3)}}{3!(n+1)^3} \\ \frac{y_n^{(4)}}{4!(n+1)^4} & \frac{y_n^{(3)}}{3!(n+1)^3} & \frac{y_n^{(2)}}{2!(n+1)^2} \\ \frac{y_n^{(3)}}{3!(n+1)^3} & \frac{y_n^{(2)}}{2!(n+1)^2} & \frac{y_n^{(1)}}{(n+1)} \end{bmatrix} \tag{27}$$

$$|A| = (n+1)18{y_n}^{(2)^2} y_n^{(5)} - 60y_n^{(2)} y_n^{(4)} y_n^{(3)} - 12y_n^{(1)} y_n^{(5)} y_n^{(3)} + 15{y_n^{(4)}}^2 y_n^{(1)} + 40{y_n^{(3)}}^3$$

$$|x_1| = \begin{bmatrix} -\frac{y_n^{(6)}}{6!(n+1)^6} & \frac{y_n^{(4)}}{4!(n+1)^4} & \frac{y_n^{(3)}}{3!(n+1)^3} \\ -\frac{y_n^{(5)}}{5!(n+1)^5} & \frac{y_n^{(3)}}{3!(n+1)^3} & \frac{y_n^{(2)}}{2!(n+1)^2} \\ -\frac{y_n^{(4)}}{4!(n+1)^4} & \frac{y_n^{(2)}}{2!(n+1)^2} & \frac{y_n^{(1)}}{(n+1)} \end{bmatrix} \tag{28}$$

$$|x_1| = 6{y_n}^{(2)^2} - 12y_n^{(1)} y_n^{(5)} y_n^{(3)} - 15{y_n^{(4)}}^2 y_n^{(2)} + 6y_n^{(1)} y_n^{(5)} y_n^{(4)} - 4y_n^{(1)} y_n^{(6)} y_n^{(3)} + 120{y_n^{(3)}}^2 y_n^{(4)}$$

$$|x_2| = \begin{bmatrix} \frac{y_n^{(5)}}{5!(n+1)^5} & -\frac{y_n^{(6)}}{6!(n+1)^6} & \frac{y_n^{(3)}}{3!(n+1)^3} \\ \frac{y_n^{(4)}}{4!(n+1)^4} & -\frac{y_n^{(5)}}{5!(n+1)^5} & \frac{y_n^{(2)}}{2!(n+1)^2} \\ \frac{y_n^{(3)}}{3!(n+1)^3} & -\frac{y_n^{(4)}}{4!(n+1)^4} & \frac{y_n^{(1)}}{(n+1)} \end{bmatrix} \tag{29}$$

$$|x_2| = 15y_n^{(2)} y_n^{(5)} y_n^{(4)} - 10y_n^{(2)} y_n^{(6)} y_n^{(3)} - 16{y_n^{(5)}}^2 y_n^{(1)} + 5y_n^{(1)} y_n^{(6)} y_n^{(4)} + 20{y_n^{(3)}}^2 y_n^{(5)} - 25{y_n^{(4)}}^2 y_n^{(3)}$$

$$|x_3| = \begin{bmatrix} \frac{y_n^{(5)}}{5!(n+1)^5} & \frac{y_n^{(4)}}{4!(n+1)^4} & -\frac{y_n^{(6)}}{6!(n+1)^6} \\ \frac{y_n^{(4)}}{4!(n+1)^4} & \frac{y_n^{(3)}}{3!(n+1)^3} & -\frac{y_n^{(5)}}{5!(n+1)^5} \\ \frac{y_n^{(3)}}{3!(n+1)^3} & \frac{y_n^{(2)}}{2!(n+1)^2} & -\frac{y_n^{(4)}}{4!(n+1)^4} \end{bmatrix} \tag{30}$$

$$|x_3| = 36{y_n}^{(5)^2} y_n^{(2)} - 30y_n^{(2)} y_n^{(4)} y_n^{(6)} - 120y_n^{(3)} y_n^{(5)} y_n^{(4)} + 75{y_n}^{(4)^3} + 40{y_n^{(3)}}^2 y_n^{(6)}$$

At the integration point $x = x_{n+1}$.

$$U(x_{n+1}) = \sum_{i=0}^{\infty} c_k\, x_{n+1}^{k}.$$

Since $q_i{}'s$ are defined by (26), we have

$$q_1 x_{n+1}^1 = \frac{h^1}{-2}\frac{\left[6{y_n}^{(2)^2} - 12y_n^{(1)}y_n^{(5)}y_n^{(3)} - 15{y_n^{(4)}}^2 y_n^{(2)} + 6y_n^{(1)}y_n^{(5)}y_n^{(4)} - 4y_n^{(1)}y_n^{(6)}y_n^{(3)} + 120{y_n^{(3)}}^2 y_n^{(4)}\right]}{\left[(n+1)18{y_n}^{(2)^2}y_n^{(5)} - 60y_n^{(2)}y_n^{(4)}y_n^{(3)} - 12y_n^{(1)}y_n^{(5)}y_n^{(3)} + 15y^{(4)^2}y_n^{(1)} + 40{y_n^{(3)}}^2\right]} \tag{31}$$

$$q_2 x_{n+1}^2 = \frac{h^2}{-10}\frac{\left[15y_n^{(2)}y_n^{(5)}y_n^{(4)} - 10y_n^{(2)}y_n^{(6)}y_n^{(3)} - 16{y_n^{(5)}}^2 y_n^{(1)} + 5y_n^{(1)}y_n^{(6)}y_n^{(4)} + 20{y_n^{(3)}}^2 y_n^{(5)} - 25{y_n^{(4)}}^2 y_n^{(3)}\right]}{\left[(n+1)18{y_n}^{(2)^2}y_n^{(5)} - 60y_n^{(2)}y_n^{(4)}y_n^{(3)} - 12y_n^{(1)}y_n^{(5)}y_n^{(3)} + 15y^{(4)^2}y_n^{(1)} + 40y^{(3)^3}\right]} \tag{32}$$

$$q_3 x_{n+1}^3 = \frac{h^3}{-120}\frac{\left[36{y_n}^{(5)^2}y_n^{(2)} - 30y_n^{(2)}y_n^{(4)}y_n^{(6)} - 120y_n^{(3)}y_n^{(5)}y_n^{(4)} + 75{y_n}^{(4)^3} + 40{y_n^{(3)}}^2 y_n^{(6)}\right]}{\left[(n+1)18{y_n}^{(2)^2}y_n^{(5)} - 60y_n^{(2)}y_n^{(4)}y_n^{(3)} - 12y_n^{(1)}y_n^{(5)}y_n^{(3)} + 15{y_n^{(4)}}^2 y_n^{(1)} + 40{y_n^{(3)}}^2\right]} \tag{33}$$

Therefore, our general integrator formula for m = 3 becomes

Now, recall from (4)

$$y_n = r + p_0$$

$$r = y_n - p_0 \tag{34}$$

$$p_1 = \frac{hy_n^{(1)}}{1!\, x_{n+1}} + p_0 q_1$$

$$p_1 x_{n+1} = hy_n^{(1)} + p_0 q_1 x_{n+1} \tag{35}$$

$$p_2 = \frac{h^2 y_n^{(2)}}{2!\, x_{n+1}^2} + \frac{hy_n^{(1)}}{1!\, x_{n+1}} + p_0 q_2$$

$$p_2 x_{n+1}^2 = \frac{h^2 y_n^{(2)}}{2!} + hy_n^{(1)} q_1 x_{n+1} + p_0 q_2 x_{n+1}^2 \tag{36}$$

$$p_0 q_3 = \frac{h^3 y_n^{(3)}}{3! x_{n+1}^3} + \frac{h^2 y_n^{(2)}}{2! x_{n+1}^2} q_1 + \frac{h^1 y_n^{(1)}}{1! x_{n+1}} q_2 \tag{37}$$

$$p_0 = \frac{-1}{q_3}\left[\frac{h^3 y_n^{(3)}}{3!} + \frac{h^2 y_n^{(2)} q_1 x_{n+1}}{2!} + hy_n^{(1)} q_2 x_{n+1}^2\right] \tag{38}$$

$$r = y_n - \frac{1}{q_3}\left[\frac{h^3 y_n^{(3)}}{3!} + \frac{h^2 y_n^{(2)} q_1 x_{n+1}}{2!} + hy_n^{(1)} q_2 x_{n+1}^2\right] \tag{39}$$

Now from (4), the rational integrator is expanded to give

$$y_{n+1}=\frac{p_0+p_1x_{n+1}+p_2x_{n+1}^2+r+rq_1x_{n+1}+rq_2x_{n+1}^2+rq_3x_{n+1}^3}{1+q_1x_{n+1}+q_2x_{n+1}^2+q_3x_{n+1}^3}. \tag{40}$$

Then,

$$p_0+r=r+p_0=y_n \tag{41}$$

$$p_1x_{n+1}+rq_1x_{n+1}=hy_n^{(1)}-\frac{q_1x_{n+1}}{q_3x_{n+1}^3}\left[\frac{h^3y_n^{(3)}}{3!}+\frac{h^2y_n^{(2)}q_1x_{n+1}}{2!}+hy_n^{(1)}q_2x_{n+1}^2\right]$$

$$+q_1x_{n+1}y_n+\frac{q_1x_{n+1}}{q_3x_{n+1}^3}\left[\frac{h^3y_n^{(3)}}{3!}+\frac{h^2y_n^{(2)}q_1x_{n+1}}{2!}+hy_n^{(1)}q_2x_{n+1}^2\right]$$

$$p_1x_{n+1}+rq_1x_{n+1}=hy_n^{(1)}+q_1x_{n+1}y_n \tag{42}$$

$$p_2x_{n+1}^2+rq_2x_{n+1}^2=\frac{h^2y_n^{(2)}}{2!}+hy_n^{(1)}q_1x_{n+1}-\frac{q_2x_{n+1}^2}{q_3x_{n+1}^3}\left[\frac{h^3y_n^{(3)}}{3!}+\frac{h^2y_n^{(2)}q_1x_{n+1}}{2!}+hy_n^{(1)}q_2x_{n+1}^2\right]$$

$$+q_2x_{n+1}^2y_n+\frac{q_2x_{n+1}^2}{q_3x_{n+1}^3}\left[\frac{h^3y_n^{(3)}}{3!}+\frac{h^2y_n^{(2)}q_1x_{n+1}}{2!}+hy_n^{(1)}q_2x_{n+1}^2\right]$$

$$p_2x_{n+1}^2+rq_2x_{n+1}^2=\frac{h^2y_n^{(2)}}{2!}+hy_n^{(1)}q_1x_{n+1}+q_2x_{n+1}^2y_n \tag{43}$$

$$rq_3x_{n+1}^3=q_3x_{n+1}^3y_n+\frac{q_3x_{n+1}^3}{q_3x_{n+1}^3}\left[\frac{h^3y_n^{(3)}}{3!}+\frac{h^2y_n^{(2)}q_1x_{n+1}}{2!}+hy_n^{(1)}q_2x_{n+1}^2\right] \tag{44}$$

Substituting equations (31) – (43) into (5), we obtain our general integrator for m = 3

$$y_{n+1}=\frac{\sum_{i=0}^{3}\frac{h^iy_n^{(i)}}{i!}+A\sum_{i=0}^{2}\frac{h^iy_n^{(i)}}{i!}+B\sum_{i=0}^{1}\frac{h^iy_n^{(i)}}{i!}+Cy_n}{1+A+B+C}. \tag{45}$$

3 Convergence, consistency and stability

For any numerical methods to be useful, such method must satisfy some basic properties. According to [8], one- step methods like a family of non linear rational integrator for the solution of ordinary differential equations are normally described symbolically by

$$y_{n+1} = y_n + h\Phi(x_n, y_n;\ h) \tag{46}$$

where $\Phi(x_n, y_n;\ h)$ is called the increment function, x_n the mesh point and h the mesh size. [9] state that a rational integrator is said to be consistent if the increment function is consistent with the initial value problem (IVP) that is if $\Phi(x, y, 0) = f(x, y)$.

Theorem 1: A non linear rational integrator of order m = 2, for the solution of ordinary

differential equations.

$$y_{n+1} = \frac{\sum_{i=0}^{2} \frac{h^i y_n^{(i)}}{i!} + A\sum_{i=0}^{1} \frac{h^i y_n^{(i)}}{i!} + By_n}{1 + A + B} \tag{47}$$

is convergent and consistent.

Proof:

Subtracting y_n from both sides of equ. (47), we have

$$y_{n+1} - y_n = \frac{\sum_{i=0}^{2} \frac{h^i y_n^{(i)}}{i!} + A\sum_{i=0}^{1} \frac{h^i y_n^{(i)}}{i!} + By_n}{1 + A + B} - y_n$$

$$y_{n+1} - y_n = \frac{\sum_{i=1}^{2} \frac{h^i y_n^{(i)}}{i!} + Ah^i y_n^{(i)}}{1 + A + B}$$

$$\frac{y_{n+1} - y_n}{h} = \frac{y_n^{(i)} + \sum_{i=2}^{3} \frac{h^{i-1} y_n^{(i)}}{i!} + Ahy_n^{(i)}}{1 + A + B + C}$$

$\lim_{h \to 0} \left(\frac{y_{n+1} - y_n}{h}\right)$: $\lim_{x \to 0} A = 0$,
$\lim_{h \to 0} \left(\frac{y_{n+1} - y_n}{h}\right) = y_n^{(1)}$,
$\lim_{h \to 0} \left(\frac{y_{n+1} - y_n}{h}\right) = y_n^{(1)} = f(x_n, y_n)$ as required.

Theorem 2: A non linear rational integrator of m = 3, for the solution of ordinary differential equations

$$y_{n+1} = \frac{\sum_{i=0}^{3} \frac{h^i y_n^{(i)}}{i!} + A\sum_{i=0}^{2} \frac{h^i y_n^{(i)}}{i!} + B\sum_{i=0}^{1} \frac{h^i y_n^{(i)}}{i!} + Cy_n}{1+A+B+C} \tag{48}$$

is convergent and consistent.

<u>Proof:</u>

Subtracting y_n from both sides of equ. (48), we have

$$y_{n+1} - y_n = \frac{\sum_{i=0}^{3} \frac{h^i y_n^{(i)}}{i!} + A\sum_{i=0}^{2} \frac{h^i y_n^{(i)}}{i!} + B\sum_{i=0}^{1} \frac{h^i y_n^{(i)}}{i!} + Cy_n}{1+A+B+C} - y_n$$

$$y_{n+1} - y_n = \frac{\sum_{i=1}^{3} \frac{h^i y_n^{(i)}}{i!} + A\sum_{i=1}^{2} \frac{h^i y_n^{(i)}}{i!} + Bh^i y_n^{(i)}}{1+A+B+C}$$

$$\frac{y_{n+1} - y_n}{h} = \frac{y_n^{(i)} + \sum_{i=2}^{3} \frac{h^{i-1} y_n^{(i)}}{i!} + A\sum_{i=1}^{2} \frac{h^{i-1} y_n^{(i)}}{i!} + Bhy_n^{(i)}}{1+A+B+C}$$

$\underset{h\to 0}{\text{Limit}}\left(\frac{y_{n+1} - y_n}{h}\right)$: $\underset{x\to 0}{\text{Limit}}\ A = \underset{x\to 0}{\text{Limit}}\ B = 0$

$\underset{h\to 0}{\text{Limit}}\left(\frac{y_{n+1} - y_n}{h}\right) = y_n^{(1)}$

$\underset{h\to 0}{\text{Limit}}\left(\frac{y_{n+1} - y_n}{h}\right) = y_n^{(1)} = f(x_n, y_n)$as required.

The Stability Analysis of the Jordan Curve at m = 2:

The Jordan curve is the equation governing the region of absolute stability and the region of instability.

We take various degrees of θ from 0^0 to 360^0 and get a corresponding values of R.

Table 1: Stability Analysis of the Jordan Curve at m = 2

θ^0	R	Approximationfor Curve Skehch	θ^0	R	Approximation for Curve Skehch
0	5.61396669241456	5.61	195	-5.32137805328698	-5.32
15	5.32137805328698	5.32	210	-5.01260658797262	-5.01
30	5.01260658797262	5.01	225	-4.83406362171214	-4.83
45	4.83406362171214	4.83	240	-4.52694187340751	-4.53
60	4.52694187340751	4.53	255	-4.21018688238941	-4.21
75	4.21018688238941	4.21	270	0.00000000000000	0

90	0.00000000000000	0	285	4.21018688238941	4.21
105	-4.21018688238941	-4.21	300	4.52694187340751	4.53
120	-4.52694187340751	-4.53	315	4.83406362171214	4.83
135	-4.83406362171214	-4.83	330	5.01260658797262	5.01
150	-5.01260658797262	-5.01	345	5.32137805328698	5.32
165	-5.32137805328698	-5.32	360	5.61396669241456	5.61
180	-5.61396669241456	-5.61			

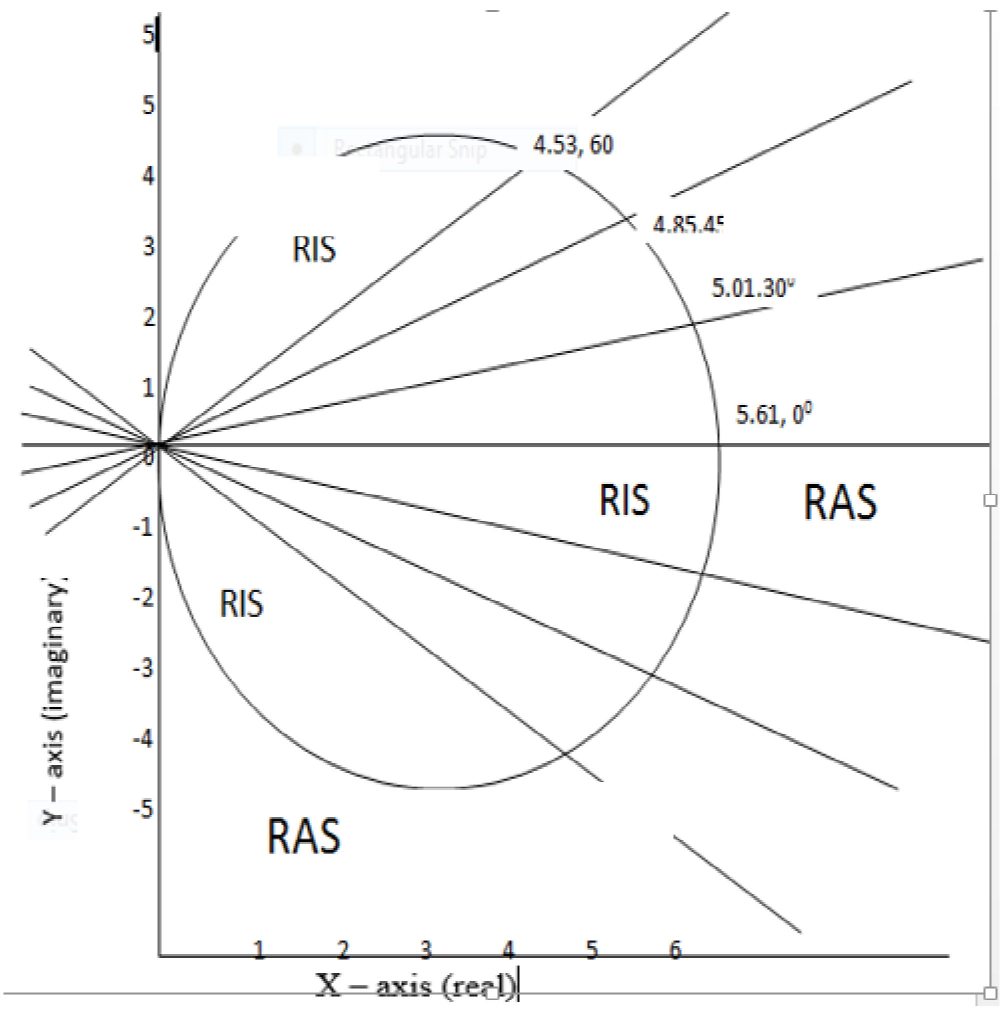

Figure 1: RAS and RIS for the case m = 2.

Hence the integrator is A-stable and so the Region of Absolute Stability of the integrator is the entire left – half of the complex plane and the space outside the diagram.

The Stability Analysis of the Jordan Curve m = 3:
The Jordan curve is the equation governing the region of absolute stability and the region of instability.
We take various degrees of θ from 0^0 to 360^0 and get a corresponding values of R.

Table 2. Stability analysis of the Jordan Curve at m = 3

θ^0	R	Approximation for Curve Skehch	θ^0	R	Approximati onfor Curve Skehch
0	7.74596669241483	7.74	195	-7.67137805328621	-7.7
15	7.67137805328621	7.7	210	-7.46260444797262	-7.5
30	7.46260444797262	7.5	225	-7.16406362171214	-7.2
45	7.16406362171214	7.2	240	-6.84694187340751	-6.9
60	6.84694187340751	6.8	255	-6.60018688238941	-6.6
75	6.60018688238941	6.6	270	0.00000000000000	0
90	0.00000000000000	0	285	6.60018688238941	6.6
105	-6.60018688238941	-6.6	300	6.84694187340751	6.9
120	-6.84694187340751	-6.9	315	7.16406362171214	7.2
135	-7.16406362171214	-7.2	330	7.46260444797262	7.5
150	-7.46260444797262	-7.5	345	7.67137805328621	7.7
165	-7.67137805328621	-7.6	360	7.74596669241483	7.74
180	-7.74596669241483	-7.74			

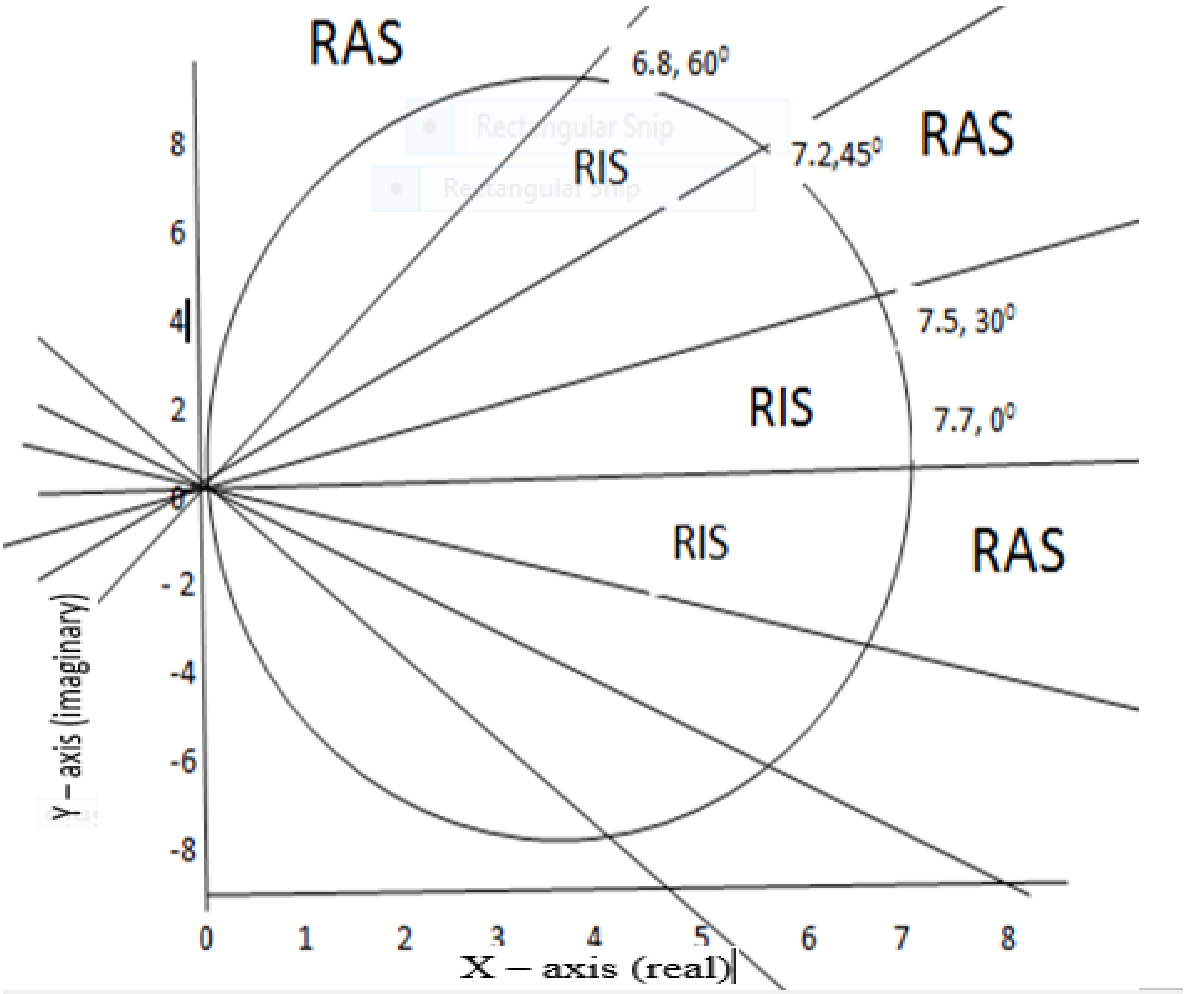

Figure 2: RAS and RIS for the case m = 3.

Hence the integrator is A-stable and so the Region of Absolute Stability of the integrator is the entire left – half of the complex plane and the space outside the diagram.

4 Numerical experiments

Problem 1: [3]

Consider $y' = y - y^2$; $y(0) = 0.5$; $h = 0.1$

The exact solution is $\frac{1}{(1+e^{-x})}$. Implicit 4th Order RKF

Table 3a: Numerical integration

XN	TSOL	Agbeboh & Esekhaigbe (2016)	New Rational Integrator of order m - 1	
			m=2	m=3
1.00E-01	5.249E-01	0.5249D+00	5.249E-01	5.249E-01
2.00E-01	5.498E-01	0.5498D+00	5.498E-01	5.498E-01
3.00E-01	5.744E-01	0.5744D+00	5.744E-01	5.744E-01
4.00E-01	5.986E-01	0.5986D+00	5.986E-01	5.986E-01
5.00E-01	6.224E-01	0.6224D+00	6.224E-01	6.224E-01
6.00E-01	6.456E-01	0.6456D+00	6.456E-01	6.456E-01
7.00E-01	6.681E-01	0.6681D+00	6.681E-01	6.681E-01
8.00E-01	6.899E-01	0.6899D+00	6.899E-01	6.899E-01
9.00E-01	7.109E-01	0.7109D+00	7.109E-01	7.109E-01
1.00E+00	7.310E-01	0.7310D+00	7.310E-01	7.310E-01

Table 3b: Error in numerical integration

XN	Agbeboh & Esekhaigbe (2016)	New Rational Integrator of order m - 1	
		m = 2	m = 3
1.00e-01	0.130D-08	-2.730e-07	-2.106e-11
2.00e-01	0.264D-08	-5.273e-07	-1.757e-10
3.00e-01	0.405D-08	-6.886e-07	-3.780e-10
4.00e-01	0.559D-08	-6.854e-07	-3.622e-10
5.00e-01	0.728D-08	-4.887e-07	-1.008e-09
6.00e-01	0.915D-08	-2.672e-07	-2.112e-07
7.00e-01	0.112D-07	-1.193e-06	-2.115e-07
8.00e-01	0.134D-07	-1.235e-05	-2.098e-07
9.00e-01	0.158D-07	-5.584e-04	-2.089e-07
1.00e+01	0.184D-07	-3.713e-03	-2.100e-07

Table 3 above shows the performance of our new numerical integrators. Our new integrator compete favorably well with the Implicit 4th Order RKF of [3], with a very high rate of convergence at m = 3.

Problem 2: [12]

Consider $y' = -\sin x - 200(y - \cos x)$; $y(0) = 0$; $h = 0.001$.
The exact solution is $y(x) = \cos x - e^{-200x}$.

Table 4a: Numerical integration

XN	TSOL	Sunday, et al (2013)	New Rational Integrator of order m - 1	
			m = 2	m = 3
1.00e-03	1.81268e-01	0.1812e-01	1.8126e-01	1.8126e-01
2.00e-03	3.29677e-01	0.3296e-01	3.2967e-01	3.2967e-01
3.00e-03	4.51183e-01	0.4511e-01	4.5118e-01	4.5118e-01
4.00e-03	5.50663e-01	0.5506e-01	5.5066e-01	5.5066e-01
5.00e-03	6.32108e-01	0.6321e-01	6.3210e-01	6.3210e-01
6.00e-03	6.98787e-01	0.6987e-01	6.9878e-01	6.9878e-01
7.00e-03	7.53378e-01	0.7533e-01	7.5337e-01	7.5337e-01
8.00e-03	7.98071e-01	0.7980e-01	7.9807e-01	7.9807e-01
9.00e-03	8.34660e-01	0.8346e-01	8.3466e-01	8.3466e-01
10.0e+03	8.64614e-01	0.8641e-01	8.6461e-01	8.6461e-01

Table 4b: Error in numerical integration

XN	Sunday, et al (2013)	New Rational Integrator of order m - 1	
		m = 2	m = 3
1.00e-03	1.8270e-07	6.4086e-05	4.1388e-09
2.00e-03	1.4085e-07	1.0493e-04	5.5915e-09
3.00e-03	5.5609e-07	1.2886e-04	5.5869e-09
4.00e-03	3.9270e-07	1.4066e-04	4.8676e-09
5.00e-03	2.2589e-07	1.4395e-04	3.8687e-09
6.00e-03	1.8561e-07	1.4142e-04	2.8328e-09
7.00e-03	1.5193e-07	1.3508e-04	1.8852e-09
8.00e-03	1.2601e-07	1.2639e-04	1.0808e-09
9.00e-03	1.1594e-07	1.1641e-04	4.3446e-10
1.00e+03	1.6619e-07	1.0589e-04	6.1043e-11

Table 4 above shows the performance of our new numerical integrator. The trend shows that for a given mesh point, the errors in numerical integrator decreases as m increases. Our new integrator competes fairly well with the Extended Block Integrator method of [12].

5 Conclusion

The use of numerical integrators in solving initial value problems have been demonstrated using a family non- linear rational integrator and have proven more effective. The numerical tables shows that our new integrator converges quickly to the analytic solution at each corresponding mesh point than its counterparts. The integrator compares favourably well with the recent work like [3] and [12].

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