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A polynomial-based collocation method for fractional order differential equation using Hermite and Two-point Taylor bases

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Abstract

Fractional differential equations (FDEs) provide a natural framework for modeling complex systems exhibiting memory and hereditary effects across diverse scientific domains. Due to the scarcity of analytical solutions for most FDEs, robust numerical techniques are essential. This study introduces a polynomial collocation scheme for linear and nonlinear FDEs with Caputo derivatives. Hermite polynomials and two-point Taylor polynomials of orders four and six serve as basis functions to approximate unknown solutions. The collocation procedure reduces the differential problem to a system of algebraic equations, from which the polynomial coefficients are determined. Accuracy is quantified through absolute error and mean absolute error, with comparisons to exact solutions and Bernstein polynomial-based results from literature. Numerical experiments and graphical results demonstrate that the proposed scheme delivers accurate and computationally effective approximations for FDEs.

Keywords and phrases: fractional differential equations; collocation method; two-point Taylor polynomial; Hermite polynomial; basis functions

1 Introduction

Over the past several decades, fractional calculus has attracted substantial research interest due to its effectiveness in representing systems with memory and hereditary dynamics. In contrast to integer-order differential equations, fractional differential equations (FDEs) employ derivatives of non-integer order, thereby offering a more accurate description of complex phenomena. Consequently, FDEs have been adopted across a broad range of scientific and engineering fields such as viscoelastic materials, control systems, signal analysis, fluid dynamics, and biological modeling, where memory-dependent processes play a critical role [1][2].

The Caputo fractional derivative has become a widely used operator for modeling practical problems because it permits initial and boundary conditions formulated with integer-order derivatives. This property makes the Caputo derivative especially appropriate for numerous physical and engineering systems, which explains its extensive use in applied research [2][3].

The broad applicability of fractional differential equations is frequently limited by the lack of closed-form analytical solutions, especially when nonlinear terms are involved. As a result, designing efficient

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numerical techniques for these equations has grown into an active area of research in applied mathematics. A variety of numerical strategies have been introduced, such as finite difference schemes, spectral methods, decomposition approaches, and collocation techniques, each offering distinct advantages and drawbacks [3][4].

Collocation methods have gained recognition as a strong numerical approach due to their straightforward implementation and computational efficiency. The method approximates the solution using selected basis functions and imposes the governing equation at specific collocation points, thereby converting the differential problem into an algebraic system. This technique has been widely adopted for solving fractional differential equations and related models. Recent literature also confirms the effectiveness of collocation-based schemes, with various basis functions being utilized to address FDEs [5][6][7].

The accuracy of collocation methods depends heavily on the choice of basis functions used to approximate the solution. Classical orthogonal polynomials such as Legendre, Chebyshev, and Bernstein polynomials have been widely used because they provide good convergence rates, numerical stability, and spectral accuracy when approximating differential equation solutions. [8][9]. Methods based on Bernstein polynomials, in particular, have shown strong stability and reliable approximation performance, which has made them a common choice for fractional integro-differential and differential equations [10][11]. Recent studies have also investigated specialized and hybrid bases, including hybrid fractional Chelyshkov function collocation and Hermite wavelet collocation for fractional PDEs [12][13]. Such approaches can better represent fractional operator behavior and improve convergence rates.

Recent work has shown that collocation methods with specialized basis functions can achieve high accuracy using few basis terms, as demonstrated by shifted Lucas polynomials for time-fractional FitzHugh-Nagumo equations [14]. A review paper notes that polynomial matrix collocation methods continue to evolve, though issues like non-smooth solutions and boundary singularities from fractional derivatives remain challenging [15]. Hermite polynomials have also been studied as basis functions for differential equations due to their orthogonality and smooth approximation, which makes them effective for problems on unbounded or semi-infinite domains. Within spectral and collocation approaches, Hermite polynomials have been applied to FDEs using operational matrices and series expansions [16][17]. Studies confirm that Hermite basis expansions work for fractional problems, including a Hermite collocation scheme for variable-coefficient FDE systems [18]. Extensions like Hermite cubic spline collocation have been developed for nonlinear and variable-order FDEs [19]. However, the combined use of two-point Taylor polynomials and Hermite polynomials as basis functions in collocation methods for fractional boundary-value problems has received limited attention.

In contrast, methods based on Taylor series remain fundamental in numerical analysis due to their simple structure and straightforward implementation. While standard Taylor expansions are typically used for local approximations, the two-point Taylor polynomial provides an improved approximation by incorporating information from two distinct points in the domain [20][21]. Despite these advantages, the two-point Taylor polynomial has been rarely employed as a basis function within collocation frameworks for fractional-order differential equations. To address this gap, the present study develops a polynomial-based collocation approach that combines Hermite and two-point Taylor polynomials as basis functions for solving fractional-order differential equations. Unlike previous work focused on single basis families, this research evaluates the performance of two different polynomial bases within a unified computational scheme. The Caputo fractional derivative definition is employed due to its compatibility with classical boundary conditions [22].

The main contribution of this work is the structured development and side-by-side comparison of Hermite and two-point Taylor polynomials as basis functions in a collocation method for fractional boundary and initial value problems. The study assesses their accuracy, convergence behavior, and computational cost for both linear and nonlinear cases. The findings clarify the performance of each basis for fractional-order models and point to the two-point Taylor polynomial as a viable alternative approximation technique for researchers working with fractional dynamic systems.

2 Preliminaries and notations

2.1 Some basic definitions

Definition 1: The Caputo fractional derivative of order $\alpha$is defined as [22].

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(u)}{(t-u)^{\alpha+1-n}} du, \tag{1}$$

Where $n-1 < \alpha \leq n$, and $n$is an integer.
Definition 2: The Hermite polynomial is defined on a given interval $-\infty \leq x \leq \infty$ with a weight function $w(x) = e^{-x^2}$ and can be generated by the formula [23].

$$H_n(x) = \sum_{r=0}^{\left[\frac{n}{2}\right]} (-1)^r \frac{n!}{r!(n-2r)!} (2x)^{n-2r}, \qquad \left[\frac{n}{2}\right] = \begin{cases} \frac{n}{2} & : n \ \ is\ even \\ \frac{n-1}{2} & ; n\ is\ odd \end{cases} \tag{2}$$

Which satisfies the recurrent relation

$$H_{n+1}(x) = 2xH_n(x) - 2xH_{n-1}(x) \tag{3}$$

and a few terms can be obtained as given in (4)

$$H_0(x) = 1, H_1(x) = (2x), H_2(x) = (4x^2 - 2), H_3(x) = (8x^3 - 12),$$

$$H_4(x) = (16x^4 - 48x^2 + 12) \tag{4}$$

The Hermite Polynomial Function (HPF) for n constant is defined as (5)

$$H_n(x) = a_0 h_0(x) + a_1 h_1(x) + \ldots + a_n h_n(x). \tag{5}$$

Where $a_0, a_1, \ldots a_n$ are constants to be determined.
Definition 3: Let a and b be two distinct points; if $f(x) \in C^{2m}\ [a, b]$ then $f(x)$ can be approximated by the polynomial $P_m(x)$ given as [24].

$$P_m(x) = (x-a)^m \sum_{k=0}^{m-1} B_k \frac{(x-b)^k}{k!} + (x-b)^m \sum_{k=0}^{m-1} A_k \frac{(x-a)^k}{k!} \tag{6}$$

where

$$A_k = \frac{d}{dx^k} \left[\frac{f(x)}{(x-b)^m} \right]_{x=a,} \quad B_k = \frac{d^k}{dx^k} \left[\frac{f(x)}{(x-a)^m} \right]_{x=b,} \tag{7}$$

and the remainder is given as

$$f(x) - P_m(x) = \frac{f^{2m}(\theta)}{(2m)!} (x-a)^m (x-b)^m \tag{8}$$

with $a < \theta < b$.
Equation (6) is referred to as a two-point Taylor polynomial approximation.
The Two-Point Taylor Polynomial (TPTP) for n constant is defined as (9)

$$\begin{aligned} T_n = {} & A_1(x-a)^n + A_2(x-a)^n(b-x) + A_3(x-a)^n(b-x)^2 + \ldots + A_n(x-a)^n(b-x)^{n-1} \\ & + B_1(x-a)(b-x)^n + B_2(x-a)^2(b-x)^n + B_3(x-a)^3(b-x)^n + \ldots + B_n(b-x)^n. \end{aligned} \tag{9}$$

Where $A_1 \ldots A_n,\ B_1 \ldots B_n$ are the constants to be determined, $n$is the order of the polynomial, a and b are the boundary points of the polynomial.

3 Methodology

In this study, the collocation minimization approach is applied to both linear and nonlinear fractional-order differential equations of the Caputo type, using HPF and TPTPF as basis functions as stated in (5) and (9).

3.1 The procedure of collocation for fractional-order differential equations involving boundary or initial conditions

The general form of a fractional-order differential equation is given as:

$$^{c}D^{\alpha}y(x)+Ry(x)+Ny(x)=f(x), \tag{10}$$

Subject to boundary conditions, $y(\alpha_1)=\beta_1$and $y(\alpha_2)=\beta_2$, $\alpha_1<x<\alpha_2$, $n-1<\alpha<n$, where $^{c}D^{\alpha}y(x)$ is the Caputo fractional derivative of a function $y(x)$, $Ry(x)$ is either a fractional derivative or an integer-order derivative remaining, $Ny(x)$is the nonlinear function, $f(x)$, is the source term, and α is the fractional order.

Applying the definition of the Caputo derivative to (10), it becomes (11).

$$\frac{1}{\Gamma(n-\alpha)}\int_0^x (x-u)^{n-\alpha-1}y^n(u)du+Ry(x)+Ny(x)=f(x). \tag{11}$$

Suppose $n=1$,then we have

$$R=\frac{1}{\Gamma(1-\delta)}\int_0^x (x-u)^{-\delta}y'(u)du+Ry(x)+Ny(x)-f(x). \tag{12}$$

Now assuming, $y(x)\approx H_n(x)$ then,

$$y(x)=H_n(x) \tag{13}$$

The assumed HPF H_n is now imposed on the boundary conditions to give

$$H_n(\alpha_1)=\beta_1 \tag{14}$$

$$H_n(\alpha_2)=\beta_2 \tag{15}$$

Equations (14) and (15) are algebraic equations involving some constants $b_i's, \quad i=0...n.$

The next step is to substitute H_n into the original differential equation (10) to generate the residual as

$$R=\frac{1}{\Gamma(n-\alpha)}\int_0^x (x-u)^{-\alpha}H_n'(u)du+R(H_n(u))+N(H_n(u))-f(x)=R(x). \tag{16}$$

Noting that, $R=R(x)$.

Collocating nodes $x_0,x_1,x_2...x_n$ are then carefully chosen as evaluating nodes in (16) within the solution domain to serve as evaluation points where the residual of the approximate solution is forced to vanish. These nodes are selected strategically rather than arbitrarily in order to enhance numerical accuracy, avoid instability, avoid the Runge phenomenon, and ensure convergence of the approximate solution. That is,

$$R(x)\,|_{x_i}=0, \tag{17}$$

This gives an additional set of algebraic equations that are solved simultaneously to obtain the constants in the basis functions. Substituting the constants into H_n gives the approximate solution to (10). The same approach of collocation will be employed for the second basis function, which is Two-Point Taylor Polynomial Function T_n, and different degrees of polynomials can be used.

For nonlinear illustrations, substitution of the approximate polynomial representation into the governing equation generated nonlinear algebraic systems in the unknown coefficients. These systems were solved numerically using MAPPLE 18 SOFTWARE's equation solver based on the Legendre integration formula.

The Caputo fractional derivatives of the polynomial basis functions were evaluated analytically using the standard fractional derivative formula for power functions given by (18) after expressing the Hermite and two-point Taylor basis functions in polynomial form.

$$D^{\alpha}x^m=\frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)}\,x^{m-\alpha} \tag{18}$$

4 Applications

Illustration 1: Consider the linear fractional boundary value problem [25].

$$u'' - xu'(x) + {}_0^cD_x^{0.5}u(x) = f(x), \tag{19}$$

with the boundary conditions $u(0) = u(1) = 0$ and.

$$f(x) = -3x^3 - 2x^2 + 8x + 2 + \frac{1}{\Gamma(0.5)}(2.67x^{1.5} + 3.2x^{2.5} - 4x^{0.5}), \tag{20}$$

The exact solution of this problem is given as (21)

$$u(x) = x^2(x+1) - 2x \tag{21}$$

Approximate solutions of the form (5) and (9) at $N = 4$ were assumed as (22) and (23).

$$H_4 = a_0 + 2a_1x + a_2(4x^2 - 2) + a_3(8x^3 - 12x) \tag{22}$$

And

$$T_4 = a_0x^2 + a_1x^2(1-x) + a_2x(1-x)^2 + a_3(1-x)^2 \tag{23}$$

The Caputo fractional derivative formula (1) was applied on (19), at $\delta = (0.5)$ and $n = 1$ and the residual was generated as (24) and (25).

$$\begin{aligned} R_{H_4} = {}& 48a_3x + 8a_2 - x(2a_1 + 8a_2x + a_3(24x^2 - 12)) + 14.44325334x^{5/2}a_3 + \\ & 6.018022224x^{3/2}a_2 + 2.256758334\sqrt{x}a_1 - 13.54055000\sqrt{x}a_3 + 3x^3 + 2x^2 \\ & -8x - 2 - 1.506386188x^{1.5} - 1.805406667x^{2.5} + 2.256758334x^{0.5} \end{aligned} \tag{24}$$

And

$$\begin{aligned} R_{T_4} = {}& 2a_0 + 2a_1(1-x) - 4a_1x - 4a_2(1-x) + 2a_2x + 2a_3 - x(2a_0x + 2a_1x(1-x) - a_1x^2 \\ & +a_2(1-x)^2 - 2a_2x(1-x) - 2a_3(1-x) - 1.805406667x^{5/2}a_1 + 1.805406667x^{5/2}a_2 \\ & +1.504505556x^{3/2}a_0 + 1.504505556x^{3/2}a_1 - 3.009011112x^{3/2}a_2 + 1.504505556x^{3/2}a_3 \\ & +1.128379167\sqrt{x}a_2 - 2.256758334\sqrt{x}a_3 + 3x^3 + 2x^2 - 8x - 2 - 1.506386188x^{1.5} - \\ & 1.805406667x^{2.5} + 2.256758334x^{0.5} \end{aligned} \tag{25}$$

Applying the boundary conditions on (22) and (23) results in systems of algebraic equations. $H_4|_{x=0} = 0$,$H_4|_{x=1} = 0$, $T_4|_{x=0} = 0$, and $T_4|_{x=1} = 0$ which implies

$$a_0 - 2a_2 = 0 \tag{26}$$

$$a_0 + 2a_1 + 2a_2 - 4a_3 = 0 \tag{27}$$

$$a_3 = 0 \tag{28}$$

$$a_0 = 0 \tag{29}$$

Equations (24) and (25) are collocated at $(N-2)$equally spaced points in the interval $[0, 1]$,that is being evaluatedR_{H_4} and R_{T_4} at $x = 0.25$and $x = 0.5$ gives the remaining systems of equations.

$$\begin{aligned} R_{H_4}|_{x=0.25} = {}& \tfrac{117}{8}a_3 + \tfrac{15}{2}a_2 - \tfrac{1}{2}a_1 - 3.159461667\sqrt{4}\,a_3 + 0.3761263890\sqrt{4}\,a_2 \\ & +0.5641895835\sqrt{4}\,a_1 - 2.944463065 \end{aligned} \tag{30}$$

$$\begin{aligned} R_{H_4}|_{x=0.5} = {}& 27a_3 + 6a_2 - a_1 - 4.964868332\sqrt{2}\,a_3 + 1.504505556\sqrt{2}\,a_2 \\ & +1.128379167\sqrt{2}\,a_1 - 4.380972646 \end{aligned} \tag{31}$$

and,

$$\begin{aligned} R_{T_4}|_{x=0.25} = {}& \tfrac{15}{8}a_0 + \tfrac{27}{64}a_1 - \tfrac{163}{64}a_2 + \tfrac{19}{8}a_3 + 0.06582211808\sqrt{4}\,a_1 + 0.1222410765\sqrt{4}\,a_2 \\ & +0.09403159725\sqrt{4}\,a_0 - 0.4701579862\sqrt{4}\,a_3 - 2.944463065 \end{aligned} \tag{32}$$

$$R_{T_4}\,|_{x=0.5} = \tfrac{3}{2}a_0 - \tfrac{9}{8}a_1 - \tfrac{7}{8}a_2 + \tfrac{5}{2}a_3 + 0.1504505556\sqrt{2}\,a_1 + 0.0376126389\sqrt{2}\,a_2 \\ +0.3761263890\sqrt{2}\,a_0 - 0.7522527780\sqrt{2}\,a_3 - 4.380972646 \tag{33}$$

In order to minimize the residual, equations (30), (31), (32), and (33) are set to zero to give an additional set of algebraic equations, which are solved alongside (26), (27), (28), and (29) to give the constant as,

$$a_0 = 0.4999662042,\ a_1 = -0.2498923164,\ a_2 = 0.2499831021,\ a_3 = 0.1250369439,$$

and,

$$a_0 = 0,\ a_1 = -3.000523512,\ a_2 = -2.000227960,\ a_3 = 0,$$

Putting the values of constants into (22) and (23) gives the required approximate solution as (34) and (35).

$$H_4 = -2.000227960x + 0.9999324084x^2 + 1.000295551x^3 \tag{34}$$

$$T_4 = -3.000523512x^2(1-x) - 2.000227960x(1-x)^2 \tag{35}$$

The same approach was applied while considering the two basis functions (HP, TPTP) at order six, and the approximate solutions were gotten as (36) and (37).

$$H_6 = 0.0000000002 - 2.000255404x + 0.9999892354x^2 + 1.000201394x^3 \\ +0.00002369520x^4 + 0.00004107904x^5 \tag{36}$$

$$T_6 = -3.000627425x^3(1-x) - 5.000735706x^3(1-x)^2 - 2.000255272x(1-x)^3 \\ -5.000777143x^2(1-x)^3 \tag{37}$$

Illustration 2: Consider the nonlinear boundary value problem of Caputo fractional order [26]

$${}_0^cD_x^{0.25}y(x) + xy^2(x) = f(x), \quad 0 < x < 1, \tag{38}$$

with the boundary conditions $y(0) = 0, y(1) = 1$
where

$$f(x) = \frac{32}{21\Gamma(0.75)}x^{1.75} + x^5, \tag{39}$$

and the exact solution is given as (40)

$$y(x) = x^2 \tag{40}$$

The present example was solved by the method of collocation using both Hermite polynomial and two-point Taylor polynomial functions of order four and six as basis functions. That is, an approximate solutions of the form (5) and (9) at $N = 4$ and $N = 6$ were assumed as (41), (42), (43) and (44).

$$H_4 = a_0 + 2a_1x + a_2(4x^2 - 2) + a_3(8x^3 - 12x) \tag{41}$$

$$H_6(x) = 0.00000000003 + 0.5489280678x - 0.4980311151x^2 - 0.09943198665x^3 \\ +0.04449400498x^4 + 0.004041029011x^5 \tag{42}$$

$$T_4 = a_0x^2 + a_1x^2(1-x) + a_2x(1-x)^2 + a_3(1-x)^2 \tag{43}$$

$$T_6(x) = 0.5472489582x^3(1-x) + 1.152794117x^3(1-x)^2 + 0.5489280679x(1-x)^3 \\ +1.148753089x^2(1-x)^3 \tag{44}$$

By applying the boundary condition and calculating Caputo derivative of $y(x)$at 0.25the residual could be generated as

$$R_{H_4} = 10.85239109x^{11/4}a_3 + 4.974012583x^{7/4}a_2 + 2.176130505x^{3/4}a_1 - 13.05678303x^{3/4}a_3 \\ +x(a_1 + 2a_1x + a_2(4x^2-2) + a_3(8x^3-12x))^2 - \tfrac{32}{21}\tfrac{x^{1.75}}{\Gamma(\frac{3}{4})} - x^5 \tag{45}$$

$$\begin{aligned} R_{H_6} = {} & 48.74056349x^{19/4}a_5 + 23.15176766x^{15/4}a_4 + 10.85239109x^{11/4}a_3 \\ & -217.0478218x^{11/4}a_5 + 4.974012583x^{7/4}a_2 - 59.68815099x^{7/4}+ \\ & 2.176130505x^{3/4}a_1 - 13.05678303x^{3/4}a_3 + 130.5678303x^{3/4}a_5+ \\ & x(a_1 + 2a_1x + a_2(4x^2-2) + a_3(8x^3-12x) + a_4(16x^4-48x^2+12) \\ & +a_5(32x^5-160x^3+120x))^2 - \tfrac{32}{21}\tfrac{x^{1.75}}{\Gamma(\frac{3}{4})} - x^5 \end{aligned} \tag{46}$$

$$\begin{aligned} R_{T_4} = {} & -1.356548886x^{11/4}a_1 + 1.356548886x^{11/4}a_2 + 1.243503146x^{7/4}a_0+ \\ & 1.243503146x^{7/4}a_1 - 2.487006292x^{7/4}a_2 + 1.243503146x^{7/4}a_3+ \\ & 1.088065252x^{3/4}a_2 - 2.176130505x^{3/4}a_3 + x(a_0x^2 + a_1x^2(1-x)+ \\ & a_2x(1-x)^2 + a_3(1-x)^2)^2 - \tfrac{32}{21}\tfrac{x^{1.75}}{\Gamma(\frac{3}{4})} - x^5 \end{aligned} \tag{47}$$

$$\begin{aligned} R_{T6} = {} & 1.523142609x^{19/4}a_2 - 1.523142609x^{19/4}a_4 - 1.446985479x^{15/4}a_1 \\ & -2.893970957x^{15/4}a_2 - 1.446985479x^{15/4}a_3 + 4.340956436x^{15/4}a_4 \\ & +1.356548886x^{11/4}a_0 + 1.356548886x^{11/4}a_2 + 4.069646659x^{11/4}a_3 \\ & -4.069646659x^{7/4}a_4 - 1.356548886x^{11/4}a_5 - 3.730509437x^{7/4}a_3 \\ & +1.243503146x^{7/4}a_4 + 3.730509437x^{7/4}a_5 + 1.088065252x^{3/4}a_3 \\ & -3.264195758x^{3/4}a_5 + x(a_0x^3 + a_1x^3(1-x) + a_2x^3(1-x)^2+ \\ & a_3x(1-x)^3 + a_4x^2(1-x)^3 + a_5(1-x)^3)^2 - \tfrac{32}{21}\tfrac{x^{1.75}}{\Gamma(\frac{3}{4})} - x^5 \end{aligned} \tag{48}$$

The same procedure as that of illustration 1 was followed, and the required approximate solutions were obtained.

$$H_4 = -0.0000000001 - 0.0202099991x + 1.092807307x^2 - 0.07259730794x^3 \tag{49}$$

$$\begin{aligned} H_6 = {} & -0.026944219x + 1.299492022x^2 - 1.183285698x^3 + 1.972338043x^4 \\ & -1.061600148x^5 \end{aligned} \tag{50}$$

$$T_4 = 1.000000000x^2 + 0.0111272946x^2(1-x) - 0.002949001113x(1-x)^2 \tag{51}$$

$$\begin{aligned} T_6 = {} & 1.000000000x^3 + 1.235931577x^3(1-x) + 0.6029475843x^3(1-x)^2 \\ & -0.004486823546x(1-x)^3 + 1.049024382x^2(1-x)^3 \end{aligned} \tag{52}$$

The computational results are summarized in the tables.
Illustration 3: Consider an inhomogeneous nonlinear fractional differential equation [27].

$$D^{\alpha}y(x) + e^{-2y(x)} = 0,\ y(0) = 0,\ y'(0) = 1, \quad 1 < \alpha \leq 2, \tag{53}$$

The exact solution, when $\alpha = 2$is $\ln(1+x)$, which has been solved in [28].
And the inhomogeneous term: $g(x) = 0$
Approximate solutions of the form (5) and (9) at $N = 4$ and $N = 6$ were also assumed and the same procedure as that of illustrations 1 and 2 were followed to obtained the required approximate solutions at $\alpha = 2$ as,

$$H_4 = 0.0000000001 + 1.000000000x - 0.4171205684x^2 + 0.1319171524x^3 \tag{54}$$

$$\begin{aligned} H_6 = {} & -0.00000000009 + 1.000000000x - 0.4731629544x^2 + 0.2537587841x^3 \\ & -0.1034020648x^4 + 0.02109103141x^5 \end{aligned} \tag{55}$$

$$T_4 = 0.7147965842x^2 + 0.8680828476x^2(1-x) + 1.000000000x(1-x)^2 \tag{56}$$

$$\begin{aligned} T_6 = {} & 0.6982847947x^3 + 1.588057045x^3(1-x) + 2.547928074x^3(1-x)^2 \\ & +1.000000000x(1-x)^3 + 2.526837039x^2(1-x)^3 \end{aligned} \tag{57}$$

The problem was solved for fractional order at $\alpha = 1.5$, and $\alpha = 1.75$ while considering , T_4, T_6, H_4 and H_6 at $N = 4$ and $N = 6$ and the results were presented in the tables.

For each of the illustrations considered, the error and mean absolute error was calculated using (58) and (59).

$$\Delta_{AE} = |x_A - x_E| \,. \tag{58}$$

$$\Delta_{MAE} = \frac{1}{n}\sum_{i}^{n} |x_A - x_E| \tag{59}$$

Where Δ_{AE}= Absolute error , Δ_{MAE} =Mean absolute error, x_A= Approximate solution and x_E= Exact solution

5 Results and Discussion

Table 1: Absolute error of illustration 1 at $N = 4$ and $\alpha = 0.5$ with HP and TPTP as basis functions in comparison with that of literature.

x	E_4(HP)	E_4(TPTP)	E_6(HP)	E_6(TPTP)	Bernstein [25]
0.1	0.00002317636	0.0000231764	0.00000315626	0.0000018651	0.104
0.2	0.00004593125	0.0000459313	0.00000561078	0.0000030572	0.132
0.3	0.0000664914	0.0000664914	0.0000063971	0.0000026071	0.204
0.4	0.0000830834	0.0000830834	0.0000046147	0.0000003767	0.235
0.5	0.0000939340	0.000093934	0.0000001650	0.0000062655	0.240
0.6	0.0000972700	0.000097270	0.0000074951	0.000014467	0.295
0.7	0.0000913179	0.000091318	0.0000156461	0.000022981	0.385
0.8	0.000074304	0.000074304	0.000021201	0.000027959	0.397
0.9	0.000044456	0.000044456	0.000018655	0.000023258	0.258
1.0	1×10^{-9}	0	1×10^{-9}	0	0
MAE	6.199653×10^{-5}	6.199645×10^{-5}	8.294204×10^{-6}	1.028366×10^{-5}	2.2500000×10^{-1}

Table 2: Absolute error of illustration 2 at $\alpha = 0.25$ for different orders of polynomials.

x	E_4(HP)	E_4(TPTP)	E_6(HP)	E_6(TPTP)
0.1	0.001165524248	0.0001387234382	$8.180800220 \times 10^{-7}$	5.4014×10^{-7}
0.2	0.0009104861035	0.0000213987131	0.00003072667984	0.00000961623
0.3	0.000329530486	0.0002675164006	0.0001944010747	0.00004581301
0.4	0.002118941672	0.0006435641277	0.0006290808274	0.0001286548
0.5	0.004022163608	0.001022286695	0.001422360342	0.0002674271
0.6	0.00560361244	0.001319226325	0.002529979350	0.0004499975
0.7	0.00642770434	0.001449925246	0.00369361341	0.0006316378
0.8	0.00605885543	0.001329925682	0.00435866440	0.0007238448
0.9	0.00406148189	0.0008747698583	0.00359205101	0.0005831617
1.0	1.4×10^{-10}	0	0	0
MAE	$3.069830036 \times 10^{-3}$	$7.067336486 \times 10^{-4}$	$1.645169517 \times 10^{-3}$	$2.840693080 \times 10^{-4}$

Table 3: Absolute error of illustration 3 at classical order ($\alpha = 2.0$) for different orders of polynomials.

x	E_4(HP)	E_4(TPTP)	E_6(HP)	E_6(TPTP)
0.1	0.00065053177	0.00065053167	0.00020182005	0.00020182009
0.2	0.0020489578	0.0020489577	0.0006233010	0.0006233010
0.3	0.0036566475	0.0036566475	0.0011162512	0.0011162509
0.4	0.0052311704	0.0052311703	0.0016311321	0.0016311317
0.5	0.0067443939	0.0067443939	0.0021604668	0.0021604663
0.6	0.0083270712	0.0083270712	0.0027087355	0.0027087348
0.7	0.0102302538	0.0102302538	0.0032790979	0.0032790970
0.8	0.0127977534	0.0127977535	0.0038711652	0.0038711640
0.9	0.0164460576	0.0164460577	0.0044862226	0.0044862214
1.0	0.0216494034	0.0216494036	0.0051376156	0.0051376141
MAE	$8.778224077 \times 10^{-3}$	$8.778224087 \times 10^{-3}$	$2.521580795 \times 10^{-3}$	$2.521580129 \times 10^{-3}$

Table 4: Result of illustration 1 at $\alpha = 0.5$ for the two polynomials of order 4 and 6 in comparison with the exact solution

x	H_4	T_4	H_6	T_6	Exact Solution
0.1	-0.1890231763	-0.1890231764	- 0.1890031562	- 0.1890018651	-0.189
0.2	-0.3520459313	-0.3520459313	- 0.3520056108	- 0.3520030572	-0.352
0.3	-0.4830664913	-0.4830664914	- 0.4830063971	- 0.4830026071	-0.483
0.4	-0.5760830834	-0.5760830834	- 0.5760046147	- 0.5759996233	-0.576
0.5	-0.6250939340	-0.6250939340	- 0.6249998352	- 0.6249937345	-0.625
0.6	-0.6240972700	-0.6240972699	- 0.6239925053	- 0.6239855333	-0.624
0.7	-0.5670913179	-0.5670913178	- 0.5669843545	- 0.5669770192	-0.567
0.8	-0.4480743045	-0.4480743042	- 0.4479787978	- 0.4479720412	-0.448
0.9	-0.2610444565	-0.2610444561	- 0.2609813450	- 0.2609767417	-0.261
1.0	-1×10^{-9}	0	$- 9.6 \times 10^{-10}$	0	0

Table 5: Result of illustration 2 at $\alpha = 0.25$ for the two polynomials of order 4 and 6 in comparison with the exact solution

x	H_4	T_4	H_6	T_6	Exact Solution
0.1	0.008834475752	0.009861276560	0.01000081808	0.01000054014	0.01
0.2	0.03908951390	0.03997860129	0.04003072667	0.04000961623	0.04
0.3	0.09032953049	0.09026751640	0.09019440108	0.09004581301	0.09
0.4	0.1621189417	0.1606435641	0.1606290809	0.1601286548	0.16
0.5	0.2540221637	0.2510222867	0.2514223603	0.2502674271	0.25
0.6	0.3656036124	0.3613192263	0.3625299793	0.3604499975	0.36
0.7	0.4964277043	0.4914499252	0.4936936133	0.4906316378	0.49
0.8	0.6460588554	0.6413299257	0.6443586645	0.6407238448	0.64
0.9	0.8140614819	0.8108747699	0.8135920511	0.8105831617	0.81
1.0	1.000000000	1.000000000	1.000000000	1.000000000	1.00

Table 6: Result of illustration 3 at H_6 for different fractional order $\alpha = 1.5$, $\alpha = 1.75$, and $\alpha = 2.0$.

x	$\alpha = 1.5$,	$\alpha = 1.75$,	$\alpha = 2.0$	Exact Solution ($\alpha = 2.0$)
0.1	0.09031222404	0.09355716110	0.09551199985	0.09531017980
0.2	0.1657421694	0.1763604870	0.1829448578	0.1823215568
0.3	0.2316184795	0.2510627340	0.2634805157	0.2623642645
0.4	0.2916494938	0.3196772127	0.3381033687	0.3364722366
0.5	0.3482499803	0.3836868276	0.4076255749	0.4054651081
0.6	0.4028678674	0.4441531157	0.4727123647	0.4700036292
0.7	0.4563109760	0.5018252858	0.5339073490	0.5306282511
0.8	0.5090737517	0.5572492582	0.5916578301	0.5877866649
0.9	0.5616639973	0.6108767031	0.6463401088	0.6418538862
1.0	0.6149296032	0.6631740805	0.6982847962	0.6931471806

Table 7: Result of illustration 3 at T_6 for different fractional order $\alpha = 1.5$, $\alpha = 1.75$, and $\alpha = 2.0$

x	$\alpha = 1.5$,	$\alpha = 1.75$,	$\alpha = 2.0$	Exact Solution ($\alpha = 2.0$)
0.1	0.09031222400	0.09355716059	0.09551199989	0.09531017980
0.2	0.1657421695	0.1763604854	0.1829448578	0.1823215568
0.3	0.2316184795	0.2510627313	0.2634805154	0.2623642645
0.4	0.2916494937	0.3196772089	0.3381033683	0.3364722366
0.5	0.3482499802	0.3836868229	0.4076255744	0.4054651081
0.6	0.4028678672	0.4441531097	0.4727123640	0.4700036292
0.7	0.4563109760	0.5018252788	0.5339073481	0.5306282511
0.8	0.5090737517	0.5572492500	0.5916578289	0.5877866649
0.9	0.5616639970	0.6108766937	0.6463401076	0.6418538862
1.0	0.6149296020	0.6631740705	0.6982847947	0.6931471806

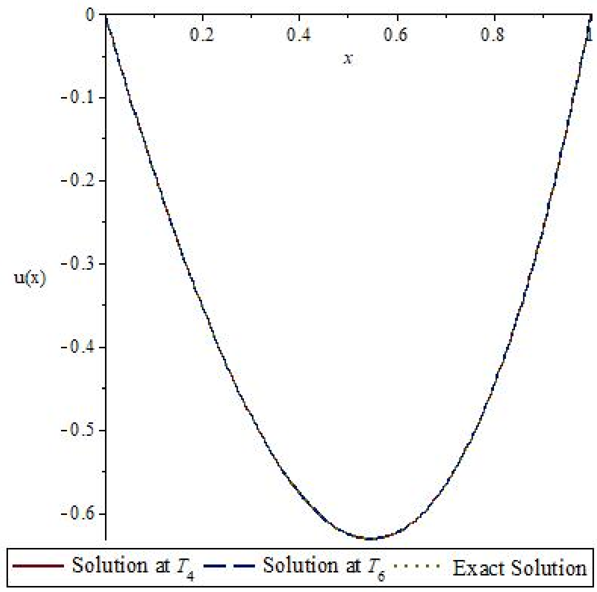

Figure 1a: Comparison of exact and approximate solutions of Illustration 1 with Hermite polynomials of orders 4 and 6 as basis functions.

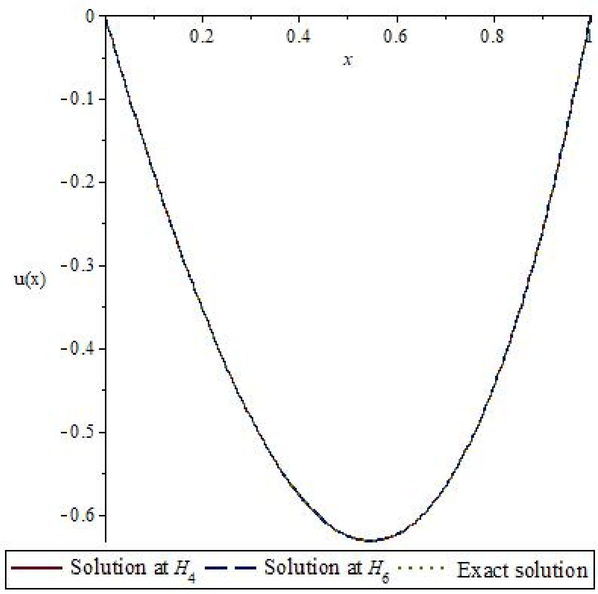

Figure 1b: Comparison of exact and approximate solutions of Illustration 1 with two-point Taylor polynomials of orders 4 and 6 as basis functions.

This section presents the numerical results obtained using the polynomial collocation framework developed for solving fractional differential equations involving the Caputo fractional derivative. Hermite polynomials (HP) and two-point Taylor polynomials (TPTP) were employed as basis functions with approximation orders four and six. Three illustrative examples comprising both linear and nonlinear fractional differential equations were considered. The performance of the method was assessed using the absolute error (AE), mean absolute error (MAE), and comparisons between the approximate and exact solutions whenever exact solutions were available.
Table 1 presents the absolute errors and MAE values obtained for Illustration 1 using HP and TPTP of orders four and six, together with the results reported in the literature using Bernstein polynomials. The results show that the present collocation framework produces significantly smaller errors than the Bernstein polynomial approach reported in [25]. The MAE values obtained are of order 10^{-5} and

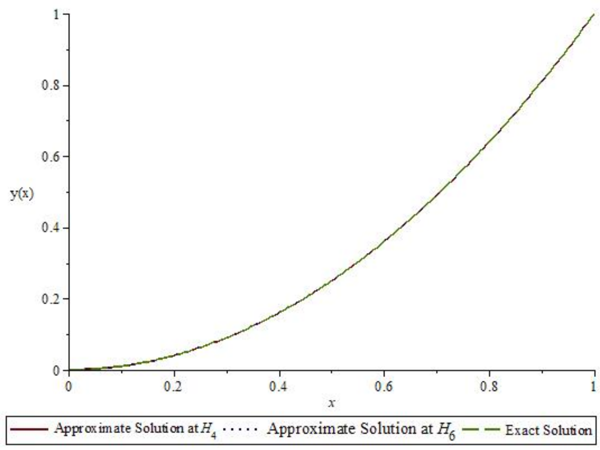

Figure 2a: Comparison of exact and approximate solutions of Illustration 2 with Hermite polynomials of orders 4 and 6 as basis functions.

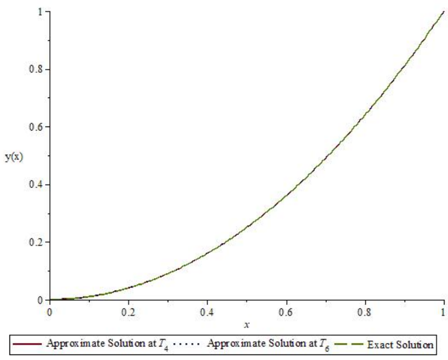

Figure 2b: Comparison of exact and approximate solutions of Illustration 2 with two-point Taylor polynomials of orders 4 and 6 as basis functions.

10^{-6}, compared with 2.2500000×10^{-1} reported in the literature, demonstrating the high accuracy of the proposed method for this problem. The closeness of the MAE values obtained using HP and TPTP indicates that both basis functions provide nearly equivalent approximations for this smooth test problem. Table 2 presents the absolute errors for Illustration 2. The results show that both polynomial bases provide accurate approximations throughout the computational domain. A comparison of the MAE values reveals that increasing the polynomial order generally improves the overall approximation accuracy. However, the reduction in pointwise errors is not entirely uniform across the interval, particularly for the nonlinear problem considered. Furthermore, the two-point Taylor polynomial exhibits lower MAE values than the Hermite polynomial for both approximation orders, indicating its superior performance for this example.

Table 3 shows the absolute errors obtained for Illustration 3 at the classical order ($\alpha = 2$). Both polynomial bases successfully approximate the exact solution, with sixth-order approximations producing substantially lower MAE values than fourth-order approximations. The errors remain relatively small across the computational interval, confirming the effectiveness of the collocation formulation for this problem.

Tables 4–7 present the approximate solutions obtained for the three illustrative examples. The numerical solutions are observed to be in close agreement with the corresponding exact solutions. For Illustration 3, additional results are presented for fractional orders ($\alpha = 1.5$) and ($\alpha = 1.75$), demonstrating the capability of the proposed method to generate approximate solutions for non-integer orders where closed-form exact solutions may not be readily available.

The graphical results shown in Figures 1a–3b support the numerical findings. In all cases, the approximate solution curves closely follow the exact solutions over the entire computational domain.

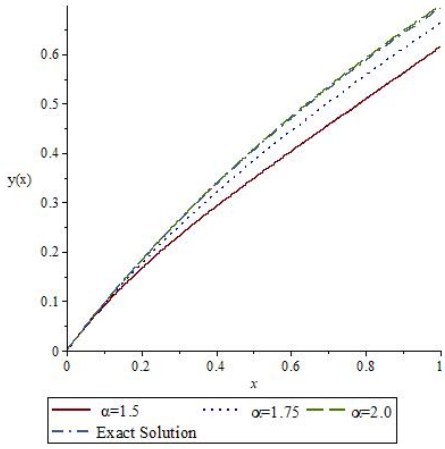

Figure 3a: Solution of illustration 3 with HP of order six as basis function at $\alpha = 1.5$, $\alpha = 1.75$, and exact solution ($\alpha = 2.0$).

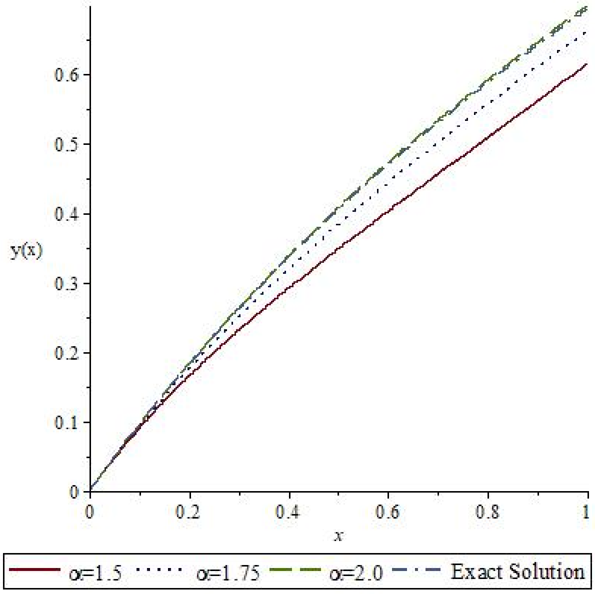

Figure 3b: Solution of illustration 3 with TPTP of order six as the basis function at $\alpha = 1.5$, $\alpha = 1.75$, and exact solution ($\alpha = 2.0$).

The agreement becomes more pronounced for the sixth-order approximations, reflecting the improved approximation capability of higher-order polynomial bases. In several instances, the approximate and exact solution curves are nearly indistinguishable.

Overall, the numerical experiments demonstrate that the proposed polynomial collocation framework provides accurate approximations for both linear and nonlinear fractional differential equations. The results further show that the performance of the basis functions is problem-dependent, with neither basis consistently outperforming the other in all cases. Nevertheless, both Hermite and two-point Taylor polynomial bases provide reliable approximations within the proposed collocation framework.

6 Conclusion

This study developed a polynomial-based collocation framework for the numerical solution of fractional differential equations involving the Caputo fractional derivative. Hermite polynomials and two-point Taylor polynomials were employed as basis functions within the collocation formulation, and the resulting schemes were applied to both linear and nonlinear illustrative problems.The numerical results obtained from the three test problems show that the proposed approach produces accurate approximations when compared with available exact solutions. Error analysis based on the absolute error and mean absolute error demonstrates that both polynomial bases are capable of generating highly accurate numerical solutions. The results also indicate that increasing the approximation order generally improves the overall accuracy of the numerical solution, although localized variations in pointwise errors may occur for some nonlinear problems.

The comparative study revealed that the relative performance of the two basis functions depends on the problem under consideration. While Hermite polynomials produced competitive results in several cases, the two-point Taylor polynomial yielded lower mean absolute errors for some examples. Consequently, both polynomial bases may be regarded as effective alternatives for constructing collocation approximations of fractional differential equations.

The overall findings demonstrate that the proposed collocation framework provides an accurate numerical technique for solving fractional differential equations involving both boundary and initial conditions. Future studies may consider the application of the framework to systems of fractional differential equations, variable-order fractional models, and multidimensional fractional problems.

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